

PHASE TRANSITIONS ON THE TOEPLITZ ALGEBRAS OF BAUMSLAG-SOLITAR SEMIGROUPS

LISA ORLOFF CLARK, ASTRID AN HUEF, AND IAIN RAEURN

ABSTRACT. Spielberg has recently shown that Baumslag-Solitar groups associated to pairs of positive integers are quasi-lattice ordered in the sense of Nica. Thus they have tractable Toeplitz algebras. Each of these algebras carries a natural dynamics. Here we construct the equilibrium states (the KMS states) for these dynamics. For inverse temperatures larger than a critical value, there is a large simplex of KMS states parametrised by probability measures on the unit circle. At the critical value, and under a mild hypothesis, there is a phase transition in which this simplex collapses to a singleton. There is a further phase transition at infinity, in the sense that there are many ground states which cannot be realised as limits of KMS states with finite inverse temperatures.

1. INTRODUCTION

Spielberg [20] has recently studied a large family of C^* -algebras which includes the C^* -algebras of higher-rank graphs [9, 17] and the boundary quotients of quasi-lattice ordered groups [15, 5]. He has also shown that the Baumslag-Solitar groups are quasi-lattice ordered with boundary quotients that are typically Kirchberg algebras, and has computed the K -theory of these boundary quotients [19].

A quasi-lattice ordered group also has a (much larger) Toeplitz algebra, and the Toeplitz algebras of groups similar to Baumslag-Solitar groups have recently been shown to exhibit interesting phase transitions. Indeed, there is nontrivial overlap¹ between the Toeplitz algebras studied in [11, 12] and the Toeplitz algebras of the Baumslag-Solitar groups studied in [19] (see [12, §9]). So one naturally wonders whether there are interesting phase transitions on the Toeplitz algebras of Baumslag-Solitar groups. Here we confirm that this is indeed the case.

Suppose that c and d are nonzero integers. The Baumslag-Solitar group $G = \text{BS}(c, d)$ is the group generated by two elements a, b subject only to the relation $ab^c = b^d a$. When c and d are positive, we consider the subsemigroup P of G generated by a and b . This semigroup defines a partial order on G : $g \leq h$ means that $g^{-1}h \in P$. Spielberg proved in [19, Theorem 2.11] that the pair (G, P) is quasi-lattice ordered in the sense of Nica [15]. The Toeplitz algebra is the C^* -subalgebra $\mathcal{T}(P)$ of $B(\ell^2(P))$ generated by a left-regular representation of P by isometries $\{T_x : x \in P\}$ (but see §2 for further discussion of our conventions). This algebra carries a natural gauge action of the circle, which we can lift to an action α of \mathbb{R} . We are interested in the KMS states of the dynamical system $(\mathcal{T}(P), \alpha)$.

We show that for inverse temperatures β larger than $\ln d$, there is a large simplex of KMS_β states parametrised by the probability measures on the circle. When d does not

Date: 11 March 2015.

This research was supported by the Marsden Fund of the Royal Society of New Zealand.

¹More precisely, the groups $\text{BS}(1, d)$ are discussed in [12, §9]. There is similar overlap between the structural results and K -theory computations in [6] and [19].

divide c , there is a phase transition at the *critical inverse temperature* $\ln d$ in which this simplex collapses to a single point, and the $\text{KMS}_{\ln d}$ state factors through the boundary quotient of [5]. The condition “ d does not divide c ” has previously occurred in Spielberg’s analysis of the groupoid model for the boundary quotient, where it is shown to be necessary and sufficient for the groupoid to be topologically principal (which he calls “essentially free”) [19, Theorem 4.9].

We begin with a section on background material: we discuss our conventions concerning quasi-lattice ordered groups and their Toeplitz algebras, and the normal form for elements of Baumslag-Solitar groups which we will use throughout. The normal form identifies a family of words in P that play a vital role in computations in G and P . We call these words “stems”, and in §3 we establish some properties of the map which sends an arbitrary element of P to its stem. In §4 we give a presentation of our Toeplitz algebra which will allow us to build Hilbert-space representations. Then in §5, we turn to KMS states. The Toeplitz algebra $\mathcal{T}(P) = C^*(\{T_x : x \in P\})$ is spanned by the elements $T_x T_y^*$, and the KMS states are the states that satisfy a commutation relation involving products of two spanning elements. In Proposition 5.1 we give a characterisation of KMS states in terms of their values on individual spanning elements. This implies, for example, that all KMS states at real inverse temperatures factor through the quotient in which the generator T_b is unitary.

Our main theorem about the KMS_β states for $\beta > \ln d$ is Theorem 6.1, and the rest of §6 is devoted to its proof. The strategy is a refinement of the one developed in [11] and [12]. To build KMS states, we exploit that all KMS states think T_b is unitary: we take a carefully chosen unitary representation W of the subgroup generated by b , and induce it to a large unitary representation $\text{Ind } W$ of G . The KMS states come from the isometric representation obtained by restricting $(\text{Ind } W)|_P$ to a suitable invariant (but not reducing!) subspace. Our results at the critical inverse temperature are in Proposition 7.1, and we show by example that they are sharp: when d divides c , there is more than one $\text{KMS}_{\ln d}$ state.

Our last main result is Theorem 8.1, where we identify the ground and KMS_∞ states of our system. This seems to be harder than in previous computations of KMS structure: ground states need not factor through the same quotient of $\mathcal{T}(P)$, and hence we cannot use induced representations. But by mimicking what happens in §6, we can build suitable isometric representations with our bare hands. We close with an appendix in which we prove that the quasi-lattice ordered group (G, P) is amenable in the sense of [15, 10]. This result is not strictly needed in the rest of the paper, but it does simplify things notationally because it implies that the Toeplitz algebra is universal for Nica-covariant representations of P (see Corollary A.7).

2. BACKGROUND

2.1. Quasi-lattice ordered groups. Suppose that G is a group and P is a subsemigroup such that $P \cap P^{-1} = \{e\}$. Then there is a partial order on G such that

$$g \leq h \iff h \in gP \iff g^{-1}h \in P.$$

This partial order is left-invariant, in the sense that $g \leq h \implies kg \leq kh$.

According to Nica [15], the pair (G, P) is a *quasi-lattice ordered group* if every pair g, h in G with a common upper bound in P has a least upper bound $g \vee h$ in P . Subsequently, Crisp and Laca showed that it suffices to check that every element $g \in G$ with an upper

bound in P has a least upper bound in P [4, Lemma 7] (and that useful lemma contains several other equivalent reformulations of the definition). We write $g \vee h < \infty$ if g and h have an upper bound in P , and $g \vee h = \infty$ otherwise.

Suppose that (G, P) is quasi-lattice ordered. We consider the Hilbert space $\ell^2(P)$ with the orthonormal basis $\{e_x : x \in P\}$ of point masses. For each $x \in P$, there is an isometry T_x on $\ell^2(P)$ such that $T_x e_y = e_{xy}$ for $y \in P$. We have $T_e = 1$ (the identity operator), and $T_x T_y = T_{xy}$. In other words, T is a homomorphism of the monoid P into the monoid of isometries on $\ell^2(P)$, and we say that T is an *isometric representation* of P . Nica observed that the representation T has the extra property

$$(2.1) \quad T_x T_x^* T_y T_y^* = \begin{cases} T_{x \vee y} T_{x \vee y}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases}$$

Now we say that an isometric representation satisfying (2.1) is *Nica covariant*. Nica covariance is equivalent to

$$(2.2) \quad T_x^* T_y = \begin{cases} T_{x^{-1}(x \vee y)} T_{y^{-1}(x \vee y)}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases}$$

A quasi-lattice ordered group (G, P) has two C^* -algebras: the *Toeplitz algebra* $\mathcal{T}(P)$ is the C^* -subalgebra of $B(\ell^2(P))$ generated by the operators $\{T_x : x \in P\}$, and the *universal C^* -algebra* $C^*(G, P)$ is generated by a universal Nica-covariant representation $i : P \rightarrow C^*(G, P)$. Nica covariance implies that every word in the $i(x)$ and their adjoints reduces to one of the form $i(x)i(y)^*$, and hence

$$C^*(G, P) = \overline{\text{span}}\{i(x)i(y)^* : x, y \in P\}.$$

The Toeplitz representation $T : P \rightarrow \mathcal{T}(P)$ induces a surjection $\pi_T : C^*(G, P) \rightarrow \mathcal{T}(P)$, and a major issue considered in [15, §4] is when π_T is an isomorphism.

Because $x \vee y = y \vee x$, Nica covariance implies that the range projections $i(x)i(x)^*$ commute with each other, and then $D := \overline{\text{span}}\{i(x)i(x)^* : x \in P\}$ is a commutative C^* -subalgebra. There is a positive norm-decreasing linear map $E : C^*(G, P) \rightarrow D$ such that $E(i(x)i(y)^*) = \delta_{x,y} i(x)i(x)^*$ (see [15, §4.2] or [10, Proposition 3.1]), and we say that (G, P) is *amenable* if E is faithful. Nica proved that if (G, P) is amenable, then the Toeplitz representation π_T is injective (see [15, §4.2] or [10, Corollary 3.9]). This implies that the Toeplitz algebra has the universal property of $(C^*(G, P), i)$, and justifies the following:

Conventions. All the quasi-lattice ordered groups (G, P) in this paper are amenable (see Theorem A.1). So it makes no difference whether we use $C^*(G, P)$ or $\mathcal{T}(P)$. We choose to write $C^*(G, P)$ for the algebra because we want to emphasise the universal property, but write T for the universal Nica-covariant representation of P in $C^*(G, P)$. We write $\mathbb{N} = \{0, 1, 2, \dots\}$.

2.2. Baumslag-Solitar groups. We fix positive integers c and d . Then the *Baumslag-Solitar group* is the group

$$G := \langle a, b : ab^c = b^d a \rangle;$$

if we want to emphasise the dependence on the numbers c, d , we write $G = \text{BS}(c, d)$. We consider the submonoid P of G generated by a and b . Spielberg proved in [19, Theorem 2.11] that (provided c and d are positive) (G, P) is quasi-lattice ordered. For the rest of this paper, (G, P) denotes one of these groups.

Following [19], we write θ for the homomorphism $\theta : G \rightarrow \mathbb{Z}$ such that $\theta(a) = 1$ and $\theta(b) = 0$, and call $\theta(g)$ the *height* of g . Baumslag-Solitar groups are examples of Higman-Neumann-Neumann extensions, and each element has a unique normal form

$$g = b^{s_0} a^{\varepsilon_1} b^{s_1} \dots b^{s_{k-1}} a^{\varepsilon_k} b^{s_k}$$

in which each ε_i is ± 1 , $0 \leq s_{i-1} < d$ when $\varepsilon_i = 1$, and $0 \leq s_{i-1} < c$ when $\varepsilon_i = -1$ (see, for example, Theorem 2.1 on [13, page 182]). For elements of P , we have $\varepsilon_i = 1$ for all k and $0 \leq s_i < d$ for all $i < k$; there is no restriction on b^{s_k} except that $s_k \geq 0$, and we have $k = \theta(x)$.

3. STEMS AND THEIR PROPERTIES

Each $x \in P$ has a unique normal form $x = b^{s_0} a b^{s_1} a \dots b^{s_{k-1}} a b^{s_k}$ with $0 \leq s_i < d$ for $i < k$ and $k = \theta(x)$. We then write

$$\text{stem}(x) := b^{s_0} a b^{s_1} a \dots b^{s_{k-1}} a$$

for the *stem* of x . We write Σ_k for the set of possible stems with height k , including $\Sigma_0 := \{e\}$; note that each Σ_k is finite with cardinality d^k .

Our constructions of KMS states will involve the Hilbert space $\bigoplus_{k \geq 0} \ell^2(\Sigma_k)$, and hence properties of stems will be important throughout the paper. In this section, we describe some of these properties.

Lemma 3.1. *Suppose that $x, y \in P$. Then $\text{stem}(x \text{stem}(y)) = \text{stem}(xy)$. If s and t satisfy $y = \text{stem}(y)b^t$ and $x \text{stem}(y) = \text{stem}(x \text{stem}(y))b^s$, then $xy = \text{stem}(xy)b^{s+t}$.*

Proof. There exists t such that $y = \text{stem}(y)b^t$, and then

$$xy = x \text{stem}(y)b^t = (\text{stem}(x \text{stem}(y))b^s)b^t \quad \text{for some } s \in \mathbb{N}.$$

On the other hand, we have $xy = \text{stem}(xy)b^r$ for some $r \in \mathbb{N}$. Since both $\text{stem}(xy)b^r$ and $\text{stem}(x \text{stem}(y))b^{s+t}$ are normal forms for xy , the uniqueness of normal forms implies that $\text{stem}(x \text{stem}(y)) = \text{stem}(xy)$ and that $r = s + t$. \square

Lemma 3.2. (a) *For all $k \in \mathbb{N}$ and $m \geq 0$, the map $h_{k,m} : x \mapsto \text{stem}(b^m x)$ is a bijection of Σ_k onto Σ_k .*

(b) *For all $k \in \mathbb{N}$, the map $x \mapsto \text{stem}(b^c a x)$ is an injection of Σ_k into Σ_{k+1} .*

Proof of part (a). We prove by induction on k that $h_{k,m}$ is a bijection for all $m \in \mathbb{N}$. The result is trivial if $k = 0$. For $k = 1$, $h_{1,m}(b^i a) = b^j a$ where $m + i = nd + j$ and $0 \leq j < d$; since addition by m modulo d is a bijection on $\{0, 1, \dots, d-1\}$, $h_{1,m}$ is a bijection. Suppose that $h_{k,m}$ is a bijection for all m . Because Σ_{k+1} is finite, it suffices to see that $h_{k+1,m}$ is one-to-one. So suppose that $h_{k+1,m}(x) = h_{k+1,m}(x')$. We can write $x = b^i a y$ for some $y \in \Sigma_k$. We now define n, j by $m + i = nd + j$ and $0 \leq j < d$, and then

$$h_{k+1,m}(b^i a y) = \text{stem}(b^{m+i} a y) = b^j a \text{stem}(b^{nc} y) = b^j a h_{k,nc}(y).$$

Doing the same for $x' = b^{i'} a y'$ shows that

$$b^j a h_{k,nc}(y) = b^{j'} a h_{k,n'c}(y') \quad \text{where } m + i = nd + j \text{ and } m + i' = n'd + j'.$$

Now uniqueness of the normal form forces $j = j'$ and $h_{k,nc}(y) = h_{k,n'c}(y')$. Since $j = j'$ and $|i - i'| < d$, we must have $i = i'$ and $n = n'$ too. Now the injectivity of $h_{k,nc}$ implies that $y = y'$ and $x = b^j a y = x'$. \square

For the proof of part (b), we separate out a calculation. Notice that it applies with $m = c$, and then gives (b) for $k = 1$. (Since $\Sigma_0 = \{e\}$, (b) is trivially true for $k = 0$.)

Lemma 3.3. *Suppose that $m \in \mathbb{N}$, $b^i a, b^j a \in \Sigma_1$, and $\text{stem}(b^m a b^i a) = \text{stem}(b^m a b^j a)$. Then $i = j$.*

Proof. Write $m = nd + m'$ with $0 \leq m' < d$. Then

$$b^m a b^i a = b^{m'} a b^{nc+i} a \quad \text{and} \quad b^m a b^j a = b^{m'} a b^{nc+j} a.$$

Now we write $nc + i = n_i d + i'$ and $nc + j = n_j d + j'$, and the hypothesis gives

$$b^{m'} a b^{i'} a = \text{stem}(b^m a b^i a) = \text{stem}(b^m a b^j a) = b^{m'} a b^{j'} a.$$

Thus the uniqueness of the normal form implies that $i' = j'$, and we have

$$i = n_i d + i' - nc = n_j d + j' - nc + (n_i - n_j)d = j + (n_i - n_j)d.$$

Now $0 \leq i, j < d$ implies that $n_i - n_j = 0$, and hence $i = j$. \square

Proof of Lemma 3.2(b). Suppose that $x, y \in \Sigma_k$ and $\text{stem}(b^c a x) = \text{stem}(b^c a y)$. We write x and y in normal form as

$$x = b^{x_0} a b^{x_1} a \cdots b^{x_{k-1}} a \quad \text{and} \quad y = b^{y_0} a b^{y_1} a \cdots b^{y_{k-1}} a,$$

and prove by induction on n that $x_i = y_i$ for $0 \leq i \leq n < k$. Since the stem of $b^c a x$ begins with the stem of $b^c a b^{x_0} a$, and similarly for $b^c a y$, Lemma 3.3 implies that $x_0 = y_0$. Suppose that we have $x_i = y_i$ for $i \leq n < k - 1$. Next we put into normal form

$$b^c a b^{x_0} a b^{x_1} a \cdots b^{x_n} a = b^{s_0} a b^{s_1} a \cdots b^{s_n} a b^m;$$

by the inductive hypothesis, we have

$$\begin{aligned} b^c a x &= b^{s_0} a b^{s_1} a \cdots b^{s_n} a b^m b^{x_{n+1}} \cdots b^{x_{k-1}} a, \quad \text{and} \\ b^c a y &= b^{s_0} a b^{s_1} a \cdots b^{s_n} a b^m b^{y_{n+1}} \cdots b^{y_{k-1}} a. \end{aligned}$$

Thus

$$\begin{aligned} b^{s_0} a b^{s_1} a \cdots b^{s_n} a \text{stem}(b^m a b^{x_{n+1}} a \cdots b^{x_{k-1}} a) &= \text{stem}(b^c a x) \\ &= \text{stem}(b^c a y) = b^{s_0} a b^{s_1} a \cdots b^{s_n} a \text{stem}(b^m a b^{y_{n+1}} a \cdots b^{y_{k-1}} a), \end{aligned}$$

and we deduce that

$$\text{stem}(b^m a b^{x_{n+1}} a \cdots b^{x_{k-1}} a) = \text{stem}(b^m a b^{y_{n+1}} a \cdots b^{y_{k-1}} a).$$

In particular, we have $\text{stem}(b^m a b^{x_{n+1}} a) = \text{stem}(b^m a b^{y_{n+1}} a)$, and Lemma 3.3 implies that $x_{n+1} = y_{n+1}$. \square

Lemma 3.4. *Let $x, y \in P$ such that $x \vee y < \infty$.*

- (a) *If $\theta(y) > \theta(x)$ then there exists $t \in \mathbb{N}$ such that $x \vee y = y b^t$.*
- (b) *If $\theta(x) = \theta(y)$ then there exists $t \in \mathbb{N}$ such that either*

$$x \vee y = x = y b^t \quad \text{or} \quad x \vee y = y = x b^t.$$

Proof. For (a), suppose $x = \text{stem}(x) b^s$ for some $s \in \mathbb{N}$. Then because $x \vee y < \infty$ and $\theta(y) > \theta(x)$, $\text{stem}(y) = \text{stem}(x) \sigma$ for some stem σ . Then $y = \text{stem}(x) \sigma b^n$ for some $n \in \mathbb{N}$. Now choose a stem τ such that $\text{stem}(b^s \tau) = \sigma$ by Lemma 3.2(a) (using that the map $h_{\theta(\sigma), s}$ is surjective and so σ must be in the image). That is, $b^s \tau = \sigma b^r$ for some $r \in \mathbb{N}$. Then

$$x \tau = \text{stem}(x) b^s \tau = \text{stem}(x) \sigma b^r.$$

Therefore

$$x \vee y = \text{stem}(x) \sigma b^{\max(n,r)} = y b^{\max(n,r)-n}$$

so if we let $t = \max(n, r) - n$, then $x \vee y = y b^t$.

For part (b), if $x \vee y < \infty$ and $\theta(y) = \theta(x)$, then putting $x \vee y$ into normal form tells us that $\text{stem}(x) = \text{stem}(y)$. The result follows. \square

4. A PRESENTATION FOR THE TOEPLITZ ALGEBRA

We want to build representations of $C^*(G, P)$. For this we use:

Proposition 4.1. *Suppose that $\pi : C^*(G, P) \rightarrow B$ is a homomorphism. Then $U := \pi(T_b)$ and $V := \pi(T_a)$ are isometries, and satisfy*

- (a) $VU^c = U^d V$;
- (b) $U^* V = U^{d-1} V U^{*c}$;
- (c) $V^* U^j V = 0$ for $1 \leq j < d$.

Conversely, if U and V are isometries in a C^ -algebra B satisfying (a), (b) and (c), then there is a Nica covariant representation $S : P \rightarrow B$ such that $S_a = V$ and $S_b = U$, and a homomorphism $\pi_{U,V} : C^*(G, P) \rightarrow B$ such that $U = \pi_{U,V}(T_b)$ and $V = \pi_{U,V}(T_a)$.*

Let $\pi : C^*(G, P) \rightarrow B$ be a homomorphism. Then $\pi \circ T$ is a Nica covariant representation of (G, P) . The relation (a) follows because $ab^c = b^d a$ in P . The relation (b) follows from Nica covariance for the pair (b, a) , which has $b \vee a = ab^c$, so that

$$U^* V = \pi(T_b^* T_a) = \pi(T_{b^{-1}(b \vee a)} T_{a^{-1}(b \vee a)}^*) = \pi(T_{b^{d-1}a}) \pi(T_{b^c})^* = U^{d-1} V U^{*c}.$$

The relation (c) is Nica covariance for $(a, b^j a)$, for which we have $a \vee b^j a = \infty$. So it remains for us to prove the converse.

Remark 4.2. The relation (c) is equivalent to saying that $\{U^j V : 0 \leq j < d\}$ is a Toeplitz-Cuntz family: in other words, the $U^j V$ are isometries satisfying

$$1 \geq \sum_{j=0}^{d-1} (U^j V)(U^j V)^*.$$

For $k \geq 1$, the stems of height k are precisely the words of length k in the alphabet $\{b^j a : 0 \leq j < d\}$, and for $\sigma = b^{j_0} a b^{j_1} a \dots b^{j_{k-1}} a$ we have

$$\pi(T_\sigma) = (U^{j_0} V)(U^{j_1} V) \dots (U^{j_{k-1}} V).$$

Thus $\{\pi(T_\sigma) : \sigma \in \Sigma_k\}$ is also a Toeplitz-Cuntz family for each $k \geq 1$.

Remark 4.3. Suppose that U, V satisfy relations (a) and (c) of Proposition 4.1 and U is unitary. Then multiplying (a) on the left by U^* and the right by U^{*c} gives (b). (This argument uses the extra relation $UU^* = 1$, so it does not work when U is just an isometry.) Our relation (a) is relation (3) in [19, Theorem 3.23]. In [19, Remark 3.24], Spielberg suggests that (3) and the Toeplitz-Cuntz relation equivalent to (c) give a presentation of his Toeplitz algebra $\mathcal{T}(G, P)$. However, we think that his $\mathcal{T}(G, P)$ is intended to be $C^*(G, P)$, and that the extra relation (b) is required for that.

The hard bit in Proposition 4.1 is proving that a pair (U, V) of isometries satisfying the relations gives us a Nica-covariant isometric representation S of P in B . It is clear how to define S : write $x \in P$ in normal form $b^{s_0} a b^{s_1} a \dots a b^{s_m}$, and define

$$S_x := U^{s_0} V U^{s_1} V \dots V U^{s_m};$$

we also set $S_e := 1$. For $x, y \in P$, the product of normal forms is not necessarily a normal form, so to see that S is multiplicative, we need to put the product

$$xy = (b^{s_0}ab^{s_1}a \cdots ab^{s_m})(b^{t_0}ab^{t_1}a \cdots ab^{t_n})$$

in normal form. However, this entails pulling any factors of the form b^{kd} in $b^{s_m}b^{t_0}$ to the right, using the relation $b^da = ab^c$ to pull any such factors across each a in turn. We can perform exactly the same calculations in

$$S_x S_y = U^{s_0} V U^{s_1} V \cdots V U^{s_m} U^{t_0} V U^{t_1} V \cdots V U^{t_n}$$

using the relation (a), arriving at the formula for S_{xy} . So S is multiplicative.

To see that S is Nica covariant, we begin with a special case. To avoid losing detail in subscripts, we say that a pair (x, y) in P is *Nica covariant* when S_x, S_y satisfy the Nica covariance relation (2.2).

Lemma 4.4. *For every $s, t \in \mathbb{N}$, the pair (b^s, ab^t) is Nica covariant.*

Proof. Write $x = b^s$ and $y = ab^t$, and write $s = (n-1)d + j$ with $1 \leq j \leq d$. Then we have

$$x \vee y = \begin{cases} ab^{nc} = xb^{d-j}a = yb^{nc-t} & \text{if } nc > t \\ y = xb^{d-j}ab^{t-nc} & \text{if } nc \leq t, \end{cases}$$

and

$$S_{x^{-1}(x \vee y)} S_{y^{-1}(x \vee y)}^* = \begin{cases} U^{d-j} V U^{*(nc-t)} & \text{if } nc > t \\ U^{d-j} V U^{t-nc} & \text{if } nc \leq t. \end{cases}$$

Next we observe that (b) implies

$$(4.1) \quad U^{*r} U^{d-1} V U^{*c} = U^{*(r+1)} V \quad \text{for every integer } r \geq 0.$$

Using this, we compute

$$\begin{aligned} S_x^* S_y &= U^{*((n-1)d+j)} V U^t \\ &= U^{*j} U^{*(n-1)d} V U^t \\ &= U^{*j} V U^{*(n-1)c} U^t \quad \text{by (4.1) with } r = d-1, n-1 \text{ times} \\ &= (U^{d-j} V U^{*c}) U^{*(n-1)c} U^t \quad \text{by (4.1) with } r = j-1, \end{aligned}$$

which is $U^{d-j} V U^{*(nc-t)}$ if $nc > t$ and $U^{d-j} V U^{t-nc}$ if $nc \leq t$. \square

The next lemma will allow us to bootstrap Lemma 4.4 to longer words.

Lemma 4.5. *Suppose that (x, y) is a Nica-covariant pair with $x \vee y < \infty$ and $\theta(x) \leq \theta(y)$. If w has the form ab^t , then (x, yw) is a Nica-covariant pair.*

Proof. We have

$$S_x^* S_{yw} = (S_x^* S_y) S_w = (S_{x^{-1}(x \vee y)} S_{y^{-1}(x \vee y)}^*) S_w.$$

The assumption $\theta(x) \leq \theta(y)$ implies that $x \vee y$ has the form $y b^s$ (see Lemma 3.4), and hence Lemma 4.4 implies that $(y^{-1}(x \vee y), w) = (b^s, w)$ is Nica covariant. Thus

$$\begin{aligned} S_x^* S_{yw} &= S_{x^{-1}(x \vee y)} (S_{y^{-1}(x \vee y)}^* S_w) \\ &= S_{x^{-1}(x \vee y)} (S_{(y^{-1}(x \vee y))^{-1}(y^{-1}(x \vee y) \vee w)} S_{w^{-1}(y^{-1}(x \vee y) \vee w)}^*) \\ &= S_{x^{-1}y(y^{-1}(x \vee y) \vee w)} S_{w^{-1}(y^{-1}(x \vee y) \vee w)}^*. \end{aligned}$$

Now we recall that the partial order on (G, P) is left invariant, and hence

$$y(y^{-1}(x \vee y) \vee w) = (yy^{-1}(x \vee y)) \vee yw = (x \vee y) \vee yw = x \vee yw$$

and

$$y^{-1}(x \vee y) \vee w = y^{-1}(x \vee y) \vee y^{-1}yw = y^{-1}((x \vee y) \vee yw) = y^{-1}(x \vee yw).$$

Thus

$$S_x^* S_{yw} = S_{x^{-1}(x \vee yw)} S_{w^{-1}y^{-1}(x \vee yw)}^* = S_{x^{-1}(x \vee yw)} S_{(yw)^{-1}(x \vee yw)}^*,$$

as required. \square

Proof of Proposition 4.1. It remains for us to prove that the representation S is Nica covariant. Suppose that $x, y \in P$. It suffices to prove (2.2) when $\theta(x) \leq \theta(y)$ (otherwise take adjoints). First we suppose that $x \vee y = \infty$. We claim that $\text{stem}(x)$ is not an initial segment of $\text{stem}(y)$. To see this, suppose to the contrary that $\text{stem}(y) = \text{stem}(x)p$ and $x = \text{stem}(x)b^t$. Then Lemma 3.2(a) implies that there is a stem q such that $b^t q$ has the form pb^s . But then $xq = \text{stem}(x)pb^s$ and y has the same stem, and we can find a common upper bound for x and y of the form $\text{stem}(x)pb^r$. Thus we have a contradiction, and the claim is proved. So there are distinct stems σ, τ in $\Sigma_{\theta(x)}$ such that x has the form $x = \sigma b^s$ and $y = \tau p$. Then because $\{S_\rho = \pi(T_\sigma) : \rho \in \Sigma_{\theta(x)}\}$ is a Toeplitz-Cuntz family (Remark 4.2), we have

$$S_x^* S_y = S_{b^s}^* S_\sigma^* S_\tau S_p = 0,$$

as required in (2.2).

Next we suppose that $x \vee y < \infty$, in which case we have $x = \sigma b^s$ for $\sigma = \text{stem}(x)$, and y has the form σw for some $w \in P$ by uniqueness of the normal form. Then

$$S_x^* S_y = S_{b^s}^* S_\sigma^* S_\sigma S_w = S_{b^s}^* S_w;$$

since left invariance of the partial order gives

$$x^{-1}(x \vee y) = b^{-s}(b^s \vee w) \text{ and } y^{-1}(x \vee y) = w^{-1}(b^s \vee w),$$

it suffices to prove the result for $x = b^s$. Now we trivially have Nica covariance for (b^s, b^r) , and Lemma 4.5 gives Nica covariance for $(b^s, b^r a b^t)$. Now an induction argument using Lemma 4.5 gives Nica covariance of (b^r, w) for all w . Thus S is Nica covariant.

The universal property of $(C^*(G, P), T)$ now gives us the homomorphism $\pi_{U,V} := \pi_S : C^*(G, P) \rightarrow B$ with the required properties. \square

5. A CHARACTERISATION OF KMS STATES

The height map θ gives a strongly continuous gauge action $\gamma : \mathbb{T} \rightarrow \text{Aut } C^*(G, P)$ such that $\gamma_z(T_x) = z^{\theta(x)} T_x$. We then define $\alpha : \mathbb{R} \rightarrow \text{Aut } C^*(G, P)$ by $\alpha_t = \gamma_{e^{it}}$, and aim to study the KMS states of the dynamical system $(C^*(G, P), \alpha)$. For $x, y \in P$ we have $\alpha_t(T_x T_y^*) = e^{it(\theta(x) - \theta(y))} T_x T_y^*$, and thus each $T_x T_y^*$ is analytic, with $\alpha_z(T_x T_y^*) = e^{iz(\theta(x) - \theta(y))} T_x T_y^*$. Since the $T_x T_y^*$ span a dense subspace of $C^*(G, P)$, it follows from [16, Proposition 8.12.3] that a state ψ of $C^*(G, P)$ is a KMS $_\beta$ state of $(C^*(G, P), \alpha)$ for some $\beta \in \mathbb{R} \setminus \{0\}$ if and only if

$$(5.1) \quad \psi((T_x T_y^*)(T_p T_q^*)) = \psi((T_p T_q^*) \alpha_{i\beta}(T_x T_y^*)) = e^{-\beta(\theta(x) - \theta(y))} \psi((T_p T_q^*)(T_x T_y^*))$$

for all $x, y, p, q \in P$.

Proposition 5.1. *Let ψ be a state on $(C^*(G, P), \alpha)$. Then ψ is a KMS_β state if and only if for all $x, y \in P$ we have*

$$(5.2) \quad \psi(T_x T_y^*) = \begin{cases} e^{-\beta\theta(x)}\psi(T_{y^{-1}x}) & \text{if } \theta(x) = \theta(y) \text{ and } x \vee y = x \\ e^{-\beta\theta(x)}\psi(T_{x^{-1}y}^*) & \text{if } \theta(x) = \theta(y) \text{ and } x \vee y = y \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose ψ is a KMS_β state on $(C^*(G, P), \alpha)$ and fix $x, y \in P$. Nica covariance of T gives

$$T_y^* T_x = \begin{cases} T_{y^{-1}(y \vee x)} T_{x^{-1}(y \vee x)}^* & \text{if } x \vee y < \infty \\ 0 & \text{if } x \vee y = \infty. \end{cases}$$

The KMS condition says

$$\psi(T_x T_y^*) = e^{-\beta\theta(x)}\psi(T_y^* T_x),$$

and hence $\psi(T_x T_y^*) = 0$ unless $x \vee y < \infty$. Applying the KMS condition again gives

$$\psi(T_x T_y^*) = e^{-\beta(\theta(x) - \theta(y))}\psi(T_x T_y^*),$$

and hence also $\psi(T_x T_y^*) = 0$ unless $\theta(x) = \theta(y)$.

Now suppose that $\theta(x) = \theta(y)$ and $x \vee y < \infty$. Then

$$\psi(T_x T_y^*) = e^{-\beta\theta(x)}\psi(T_{y^{-1}(x \vee y)} T_{x^{-1}(x \vee y)}^*),$$

and we recover (5.2) since either $x \vee y = x$ or $x \vee y = y$ by Lemma 3.4(b).

Conversely, suppose ψ is a state satisfying (5.2). We fix $x, y, p, q \in P$ and aim to show the KMS condition (5.1) holds. We will show that if $\psi(T_x T_y^* T_p T_q^*) \neq 0$, then $\psi(T_p T_q^* T_x T_y^*) \neq 0$ also and the KMS condition holds. Then, by symmetry, $\psi(T_x T_y^* T_p T_q^*) \neq 0$ if and only if $\psi(T_p T_q^* T_x T_y^*) \neq 0$, and so if $\psi(T_x T_y^* T_p T_q^*) = 0$ then the KMS condition holds with both sides zero. So we assume that $\psi(T_x T_y^* T_p T_q^*) \neq 0$.

By Nica covariance $y \vee p < \infty$ and

$$(5.3) \quad 0 \neq \psi(T_x T_y^* T_p T_q^*) = \psi(T_{xy^{-1}(y \vee p)} T_{qp^{-1}(y \vee p)}^*).$$

We now argue that it suffices to show the KMS condition when $\theta(y) \geq \theta(p)$ and $y \vee p = yb^m$ for some $m \in \mathbb{N}$. If $\theta(y) > \theta(p)$, then there exists $m \in \mathbb{N}$ such that $y \vee p = yb^m$ by Lemma 3.4(a). If $\theta(y) = \theta(p)$, then there exists $m \in \mathbb{N}$ such that $y \vee p = yb^m$ by Lemma 3.4(b) ($m = 0$ is allowed). If $\theta(y) < \theta(p)$, then we take the adjoint of $T_x T_y^* T_p T_q^*$ and use that $\psi(a^*) = \overline{\psi(a)}$. So we assume that $\theta(y) \geq \theta(p)$ and $y \vee p = yb^m$ for some $m \in \mathbb{N}$.

Set

$$M := xy^{-1}(y \vee p) = xy^{-1}yb^m = xb^m \quad \text{and} \quad N := qp^{-1}(y \vee p) = qp^{-1}yb^m.$$

Then (5.3) and the equation for ψ at (5.2) implies that $\theta(M) = \theta(N)$, and either $M \vee N = M$ or $M \vee N = N$. Thus

$$\theta(x) - \theta(y) = \theta(q) - \theta(p),$$

and then $\theta(x) \geq \theta(q)$. By Lemma 3.4(b) there exists $n \in \mathbb{Z}$ such that $M = Nb^n$. For future use we note here that $M = Nb^n$ implies

$$(5.4) \quad x = qp^{-1}yb^n.$$

Using (5.2) we have

$$(5.5) \quad \psi(T_x T_y^* T_p T_q^*) = \psi(T_M T_N^*) = \begin{cases} e^{-\beta\theta(N)} \psi(T_{b^n}) & \text{if } n \geq 0 \\ e^{-\beta\theta(N)} \psi(T_{b^{-n}}^*) & \text{if } n < 0. \end{cases}$$

Next we consider $\psi(T_p T_q^* T_x T_y^*)$. Since $x \leq M$ and $q \leq N$ we have $x \vee q \leq M \vee N < \infty$, and $\psi(T_p T_q^* T_x T_y^*) = \psi(T_{pq^{-1}(x \vee q)} T_{yx^{-1}(x \vee q)}^*)$. By Lemma 3.4, there exists $s \in \mathbb{N}$ such that either $x \vee q = xb^s$ or $x \vee q = qb^s$. First, suppose that $x \vee q = xb^s$. Then

$$yx^{-1}(x \vee q) = yb^s \text{ and } pq^{-1}(x \vee q) = pq^{-1}xb^s = yb^n b^s = yx^{-1}(x \vee q)b^n$$

using (5.4). Second, suppose that $x \vee q = qb^s$. Since $\theta(x) \geq \theta(q)$, we have $x = x \vee q$. Thus

$$pq^{-1}(x \vee q) = pb^s \text{ and } yx^{-1}(x \vee q) = y = pq^{-1}xb^{-n} = pb^s b^{-n} = pq^{-1}(x \vee q)b^{-n}$$

using (5.4). In either case, $pq^{-1}(x \vee q) = yx^{-1}(x \vee q)b^n$, and

$$\psi(T_p T_q^* T_x T_y^*) = \psi(T_{pq^{-1}(x \vee q)} T_{yx^{-1}(x \vee q)}^*) = \begin{cases} e^{-\beta\theta(pq^{-1}(x \vee q))} \psi(T_{b^n}) & \text{if } n \geq 0 \\ e^{-\beta\theta(pq^{-1}(x \vee q))} \psi(T_{b^{-n}}^*) & \text{if } n < 0. \end{cases}$$

But $\theta(pq^{-1}(x \vee q)) = \theta(p) - \theta(q) + \theta(x) = \theta(y)$ and $\theta(N) = \theta(x)$. Thus

$$e^{-\beta(\theta(x) - \theta(y))} \psi(T_p T_q^* T_x T_y^*) = \begin{cases} e^{-\beta\theta(x)} \psi(T_{b^n}) & \text{if } n \geq 0 \\ e^{-\beta\theta(x)} \psi(T_{b^{-n}}^*) & \text{if } n < 0 \end{cases}$$

is the same as (5.5), as required. As we argued above, this suffices to show that ψ is a KMS_β state. \square

Corollary 5.2. *Suppose that ϕ and ψ are KMS_β states on $(C^*(G, P), \alpha)$ and $\phi(T_{b^t}) = \psi(T_{b^t})$ for all $t \in \mathbb{N}$. Then $\phi = \psi$.*

Proof. Both states vanish on generators $T_x T_y^*$ unless $\theta(x) = \theta(y)$ and $x \vee y$ is x or y , in which case Lemma 3.4 implies that either $y^{-1}x$ or $x^{-1}y$ has the form b^t . Thus $\phi(T_x T_y^*) = \psi(T_x T_y^*)$ for all $x, y \in P$, and $\phi = \psi$. \square

Corollary 5.3. *Consider the dynamical system $(C^*(G, P), \alpha)$ as above and take $\beta \in \mathbb{R}$.*

- (a) *Every KMS_β state of $(C^*(G, P), \alpha)$ factors through the quotient by the ideal generated by $1 - T_b T_b^*$.*
- (b) *If $\beta < \ln d$, then $(C^*(G, P), \alpha)$ has no KMS_β states.*
- (c) *Let I be the ideal generated by the element*

$$1 - \sum_{j=0}^{d-1} T_{b^j a} T_{b^j a}^*.$$

Then a KMS_β state factors through the quotient $\mathcal{O}(G, P) := C^(G, P)/I$ if and only if $\beta = \ln d$.*

Proof. For (a), suppose that ψ is a KMS_β state of $(C^*(G, P), \alpha)$. Then

$$\psi(T_b T_b^*) = \psi(T_b^* \alpha_{i\beta}(T_b)) = \psi(T_b^* T_b) = \psi(1) = 1.$$

Thus $\psi(1 - T_b T_b^*) = 0$. The projection $1 - T_b T_b^*$ is invariant for the dynamics, and the elements $T_x T_y^*$ are analytic elements such that $\alpha_z(T_x T_y^*)$ is the product of $T_x T_y^*$ by the scalar-valued function $z \mapsto e^{iz(\theta(x) - \theta(y))}$. So we apply Lemma 2.2 of [7] with $P = \{1 - T_b T_b^*\}$ and $\mathcal{F} = \{T_x T_y^*\}$, and deduce that ψ factors through a state of the quotient, as claimed.

For (b), we again suppose that ψ is a KMS_β state of $(C^*(G, P), \alpha)$. Then since

$$\{T_{b^j a} : 0 \leq j < d\}$$

is a Toeplitz-Cuntz family, we have $1 \geq \sum_{j=0}^{d-1} T_{b^j a} T_{b^j a}^*$ in $C^*(G, P)$. Thus (5.2) gives

$$(5.6) \quad 1 = \psi(1) \geq \sum_{j=0}^{d-1} \psi(T_{b^j a} T_{b^j a}^*) = \sum_{j=0}^{d-1} e^{-\beta} \psi(1) = e^{-\beta} d,$$

which is equivalent to $\beta \geq \ln d$.

We now prove (c). If ψ is a $\text{KMS}_{\ln d}$ state, then $1 = e^{-\beta} d$ forces equality throughout (5.6), and

$$(5.7) \quad \psi\left(1 - \sum_{j=0}^{d-1} T_{b^j a} T_{b^j a}^*\right) = 0.$$

Now another application of [7, Lemma 2.2] shows that ψ factors through the quotient $\mathcal{O}(G, P)$. Conversely, if ψ is a KMS_β state which factors through the quotient, then ψ satisfies (5.7), we have equality in (5.6), and $\beta = \ln d$. \square

6. KMS STATES FOR LARGE INVERSE TEMPERATURES

Theorem 6.1. *Suppose that $\beta > \ln d$ and μ is a probability measure on \mathbb{T} . Then there is a KMS_β state $\psi_{\beta, \mu}$ on $(C^*(G, P), \alpha)$ such that*

$$(6.1) \quad \psi_{\beta, \mu}(T_{b^t}) = (1 - e^{-\beta} d) \left(\int_{\mathbb{T}} z^t d\mu(z) + \sum_{\{k \geq 1 : d \mid c^j d^{-j} t \text{ for } 0 \leq j < k\}} e^{-\beta k} d^k \int_{\mathbb{T}} z^{c^k d^{-k} t} d\mu(z) \right),$$

where the sum is interpreted as 0 if there are no such integers k . Every KMS_β state has this form. If d does not divide c , then the map $\mu \mapsto \psi_{\beta, \mu}$ is an affine continuous isomorphism of the simplex $P(\mathbb{T})$ of probability measures on \mathbb{T} onto the KMS_β simplex of $(C^*(G, P), \alpha)$.

The proof of this theorem will occupy the rest of this section. Our first task is to show existence of such states, and for this we need some concrete representations of $C^*(G, P)$.

We consider the subgroup $K := \{b^t : t \in \mathbb{Z}\}$ of the Baumslag-Solitar group $G = BS(c, d)$, and let $W : K \rightarrow U(H)$ be a unitary representation. We choose a section $c : G/K \rightarrow G$ for the quotient map, and write $c(g)$ for $c(gK)$. Then we can realise the induced representation $\text{Ind}_K^G W$ as acting in the space $\ell^2(G/K, H)$ according to the formula

$$((\text{Ind}_K^G W)_l \xi)(gK) = W_{c(g)^{-1} l c(l^{-1} g)}(\xi(l^{-1} gK)) \quad \text{for } l \in G, \xi \in \ell^2(G/K, H).$$

(See, for example, [8, page 50].)

Proposition 6.2. *Choose a section $c : G/K \rightarrow G$ such that $c(xK) = \text{stem}(x)$ for every $x \in P$, and use c to pull over the usual orthonormal basis $\{e_{k,\sigma} : \sigma \in \Sigma_k\}$ for $\ell^2(\Sigma_k)$ to an orthonormal set in $\ell^2(G/K)$. Then the subspace*

$$H_0 := \overline{\text{span}}\{e_{k,\sigma} \otimes h : k \in \mathbb{N}, \sigma \in \Sigma_k, h \in H\}$$

of $\ell^2(G/K, H)$ is invariant for every $(\text{Ind}_K^G W)_x$ with $x \in P$, and we then have

$$(6.2) \quad (\text{Ind}_K^G W)_x(e_{k,\sigma} \otimes h) = e_{k+\theta(x), \text{stem}(x\sigma)} \otimes W_{b^t} h \quad \text{where } x\sigma = \text{stem}(x\sigma)b^t.$$

The map $x \mapsto (\text{Ind}_K^G W)_x|_{H_0}$ is a Nica-covariant isometric representation of P , and the corresponding representation π of $C^(G, P)$ on H_0 satisfies*

$$(6.3) \quad \pi(T_x)(e_{k,\sigma} \otimes h) = e_{k+\theta(x), \text{stem}(x\sigma)} \otimes W_{b^t} h \quad \text{where } x\sigma = \text{stem}(x\sigma)b^t.$$

The operator $\pi(T_b)$ is unitary.

Proof. It suffices to check invariance for the generators $x = a$ and $x = b$ of P . First we take $x = a$. We are viewing functions in $\ell^2(\Sigma_k)$ as functions on G/K by viewing Σ_k as the subset $\{\sigma K : \sigma \in \Sigma_k\}$ of G/K and extending the functions to be 0 off Σ_k . Thus

$$e_{k,\sigma}(gK) = \begin{cases} 1 & \text{if } gK = \sigma K \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} (\text{Ind}_K^G W)_a(e_{k,\sigma} \otimes h)(gK) &= W_{c(g)^{-1}ac(a^{-1}g)}(e_{k,\sigma}(a^{-1}gK)h) \\ &= \begin{cases} W_{c(g)^{-1}ac(a^{-1}g)}h & \text{if } a^{-1}gK = \sigma K \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Now we have

$$a^{-1}gK = \sigma K \iff a^{-1}g = \sigma b^n \text{ for some } n \iff g = a\sigma b^n \text{ for some } n;$$

then, since $a\sigma$ is a stem, and $c(\tau K) = \tau$ for $\tau \in \bigcup_k \Sigma_k$, we have $c(g)^{-1}ac(a^{-1}g) = (a\sigma)^{-1}a\sigma = e$. So

$$(6.4) \quad \begin{aligned} (\text{Ind}_K^G W)_a(e_{k,\sigma} \otimes h)(gK) &= \begin{cases} h & \text{if } gK = a\sigma K \\ 0 & \text{otherwise} \end{cases} \\ &= (e_{k+1,a\sigma} \otimes h)(gK). \end{aligned}$$

This gives invariance of H_0 for $(\text{Ind}_K^G W)_a$.

For $x = b$, similar considerations give

$$(\text{Ind}_K^G W)_b(e_{k,\sigma} \otimes h)(gK) = \begin{cases} W_{c(g)^{-1}bc(b^{-1}g)}h & \text{if } b^{-1}gK = \sigma K \\ 0 & \text{otherwise,} \end{cases}$$

and

$$b^{-1}gK = \sigma K \iff b^{-1}g = \sigma b^n \text{ for some } n \iff g = b\sigma b^n \text{ for some } n.$$

While $b\sigma$ need not be a stem, it is certainly in P , and hence $c(b\sigma K) = \text{stem}(b\sigma)$. Then $b\sigma = \text{stem}(b\sigma)b^t$ for some $t \in \mathbb{N}$, and

$$c(g)^{-1}bc(b^{-1}g) = (\text{stem}(b\sigma))^{-1}b\sigma = b^t.$$

Thus

$$(6.5) \quad (\text{Ind}_K^G W)_b(e_{k,\sigma} \otimes h) = e_{k,\text{stem}(b\sigma)} \otimes W_{b^t}h,$$

which is back in $\ell^2(\Sigma_k) \otimes H \subset H_0$.

Since both $U := (\text{Ind}_K^G W)_b|_{H_0}$ and $V := (\text{Ind}_K^G W)_a|_{H_0}$ are the restrictions of unitary operators, they are isometries. We next prove (6.2). First, we prove it by induction on i for x of the form b^i . It is by definition true for $i = 1$. If it is true for i , then

$$\begin{aligned} & (\text{Ind}_K^G W)_{b^{i+1}}(e_{k,\sigma} \otimes h) \\ &= (\text{Ind}_K^G W)_{b^i}(e_{k,\text{stem}(b\sigma)} \otimes W_{b^t}h) \quad \text{where } b\sigma = \text{stem}(b\sigma)b^t \\ &= e_{k,\text{stem}(b^i \text{stem}(b\sigma))} \otimes W_{b^s}W_{b^t}h \quad \text{where } b^i \text{stem}(b\sigma) = \text{stem}(b^i \text{stem}(b\sigma))b^s. \end{aligned}$$

Lemma 3.1 implies that $\text{stem}(b^{i+1}\sigma) = \text{stem}(b^i \text{stem}(b\sigma))$ and $b^{i+1}\sigma = \text{stem}(b^{i+1}\sigma)b^{s+t}$, and we have proved the result for $x = b^i$. It is quite easy to check that it works for $x = b^i a$, and then a similar argument by induction on the height of x (again using Lemma 3.1) gives the general result.

We aim to prove that U and V satisfy the relations of Proposition 4.1. Since we know from Lemma 3.2 that $\sigma \mapsto \text{stem}(b\sigma)$ is a bijection, Equation (6.5) implies that U is surjective. Thus U is unitary, and it suffices to verify that U and V satisfy relations (a) and (c) of Proposition 4.1.

For (a) we have on the one hand

$$\begin{aligned} VU^c(e_{k,\sigma} \otimes h) &= V(e_{k,\text{stem}(b^c\sigma)} \otimes W_{b^t}h) \quad \text{where } b^c\sigma = \text{stem}(b^c\sigma)b^t \\ &= e_{k+1,a \text{stem}(b^c\sigma)} \otimes W_{b^t}h \quad \text{where } b^c\sigma = \text{stem}(b^c\sigma)b^t; \end{aligned}$$

on the other hand

$$U^d V(e_{k,\sigma} \otimes h) = e_{k+1,\text{stem}(b^d a\sigma)} \otimes W_{b^s}h \quad \text{where } b^d a\sigma = \text{stem}(b^d a\sigma)b^s.$$

Now Lemma 3.1 gives $a \text{stem}(b^c\sigma) = \text{stem}(ab^c\sigma) = \text{stem}(b^d a\sigma)$, and

$$b^t = \text{stem}(b\sigma)^{-1}b^c\sigma = (a \text{stem}(b\sigma))^{-1}ab^c = \text{stem}(b^d a\sigma)^{-1}b^d a\sigma = b^s,$$

and we have (a). For (c), we observe that $U^j V(e_{k,\sigma} \otimes h)$ has the form $e_{k+1,\text{stem}(b^j a\sigma)} \otimes W_{b^t}h$. For $0 \leq j < d$ and $\sigma \in \Sigma_k$, $b^j a\sigma$ is itself a stem, and for $1 \leq j < d$ it is distinct from $a\sigma$. Thus for $1 \leq j < d$ the vector $U^j V(e_{k,\sigma} \otimes h)$ is always orthogonal to $V(e_{k,\sigma} \otimes h) = e_{k+1,a\sigma} \otimes h$, and hence to the range of V . Thus $V^* U^j V(e_{k,\sigma} \otimes h)$ is always 0, and we have (c).

Now Proposition 4.1 implies that $S : x \mapsto (\text{Ind}_K^G W)_x|_{H_0}$ is Nica covariant, and gives a homomorphism $\pi = \pi_S$ such that $\pi(T_{b^t}) = S_{b^t} = U^t$ is unitary. \square

Proposition 6.3. *In the situation of Proposition 6.2, fix a unit vector $h \in H$. Then for every $\beta > \ln d$ there is a KMS_β state ψ_h on $(C^*(G, P), \alpha)$ such that*

$$(6.6) \quad \psi_h(a) = (1 - e^{-\beta}d) \sum_{k=0}^{\infty} \sum_{\sigma \in \Sigma_k} e^{-\beta k} (\pi(a)(e_{k,\sigma} \otimes h) | e_{k,\sigma} \otimes h) \quad \text{for } a \in C^*(G, P).$$

Proof. We begin by checking that the series converges. Indeed, since $|\Sigma_k| = d^k$, and $e^\beta > d$, we have

$$\sum_{k=0}^{\infty} \sum_{\sigma \in \Sigma_k} e^{-\beta k} (e_{k,\sigma} \otimes h | e_{k,\sigma} \otimes h) = \sum_{k=0}^{\infty} e^{-\beta k} d^k = \frac{1}{1 - e^{-\beta}d}.$$

In particular, $\psi_h(1) = 1$, and we have a well-defined state.

We now want to verify that ψ_h satisfies Equation (5.2) in Proposition 5.1. So we take $x, y \in P$ and consider

$$(6.7) \quad (\pi(T_x T_y^*)(e_{k,\sigma} \otimes h) | e_{k,\sigma} \otimes h) = (\pi(T_y^*)(e_{k,\sigma} \otimes h) | \pi(T_x^*)(e_{k,\sigma} \otimes h)).$$

For each $k \geq 0$, we identify

$$\overline{\text{span}}\{e_{k,\sigma} \otimes g : \sigma \in \Sigma_k, g \in H\}$$

with $\ell^2(\Sigma_k) \otimes H$. Then the subspaces $\{\ell^2(\Sigma_k) \otimes H : k \in \mathbb{N}\}$ are mutually orthogonal, and we have

$$H_0 = \bigoplus_{k=0}^{\infty} \ell^2(\Sigma_k) \otimes H.$$

The operator $\pi(T_x)$ maps each summand $\ell^2(\Sigma_k) \otimes H$ into $\ell^2(\Sigma_{k+\theta(x)}) \otimes H$, and hence the adjoint T_x^* vanishes on $\ell^2(\Sigma_k) \otimes H$ for $k < \theta(x)$, and maps the other $\ell^2(\Sigma_k) \otimes H$ into $\ell^2(\Sigma_{k-\theta(x)}) \otimes H$. Thus when $\theta(x) \neq \theta(y)$, T_x^* and T_y^* map $\ell^2(\Sigma_k) \otimes H$ into orthogonal summands in $H_0 = \bigoplus \ell^2(\Sigma_k) \otimes H$. Thus if $\theta(x) \neq \theta(y)$, we have $\psi_h(T_x T_y^*) = 0$.

It remains to consider x, y satisfying $\theta(x) = \theta(y)$. Then by Lemma 3.4 we have one of $x \vee y = x$, $x \vee y = y$ or $x \vee y = \infty$. If $x \vee y = \infty$, then Nica covariance of T implies that $T_x^* T_y = 0$, so that the range of $\pi(T_x)$ is orthogonal to the range of $\pi(T_y)$; since $\pi(T_x)^*(e_{k,\sigma} \otimes h) = 0$ unless $e_{k,\sigma} \otimes h$ is in the range of $\pi(T_x)$, all the inner products (6.7) are 0, and $\psi(T_x T_y^*) = 0$. Since

$$(\pi(T_y T_x^*)(e_{k,\sigma} \otimes h) | e_{k,\sigma} \otimes h) = \overline{(\pi(T_x T_y^*)(e_{k,\sigma} \otimes h) | e_{k,\sigma} \otimes h)},$$

it remains for us to compute $\psi(T_x T_y^*)$ when $x \vee y = x$ (if $x \vee y = y$ switch x and y). So suppose $x \vee y = x$. Then Lemma 3.4 implies that $x = y b^t$ for some $t \in \mathbb{N}$.

We begin by fixing $\sigma \in \Sigma_k$ and computing

$$(6.8) \quad \begin{aligned} (\pi(T_x T_y^*)(e_{k,\sigma} \otimes h) | e_{k,\sigma} \otimes h) &= (\pi(T_y T_{b^t}^*)(e_{k,\sigma} \otimes h) | e_{k,\sigma} \otimes h) \\ &= (\pi(T_{b^t})\pi(T_y^*)(e_{k,\sigma} \otimes h) | \pi(T_y^*)(e_{k,\sigma} \otimes h)). \end{aligned}$$

Notice that because $\pi(T_b)$ is unitary, the operator $\pi(T_x T_y^*)$ is not changed if we replace y by its stem. So we assume that y is a stem. Then $y\sigma$ is also a stem, so the formula (6.3) implies that $\pi(T_y)(e_{k,\sigma} \otimes h) = e_{k+\theta(y),y\sigma} \otimes h$. Since each $e_{k,\sigma} \otimes h$ is either in the range of $\pi(T_y)$ or orthogonal to it, $\pi(T_y)^*(e_{k,\sigma} \otimes h)$ vanishes unless $k \geq \theta(y)$ and σ has the form $y\tau$ for some $\tau \in \Sigma_{k-\theta(y)}$. Then

$$(\pi(T_x T_y^*)(e_{k,\sigma} \otimes h) | e_{k,\sigma} \otimes h) = (\pi(T_{b^t})(e_{k-\theta(y),\tau} \otimes h) | e_{k-\theta(y),\tau} \otimes h).$$

Next we observe that because y is a stem, $\tau \mapsto y\tau$ is an injection of Σ_j into $\Sigma_{j+\theta(y)}$ for every $j \geq 0$. Thus

$$\begin{aligned} \psi_h(T_x T_y^*) &= (1 - e^{-\beta} d) \sum_{k=\theta(y)}^{\infty} \sum_{\tau \in \Sigma_{k-\theta(y)}} e^{-\beta k} (\pi(T_{b^t})(e_{k-\theta(y),\tau} \otimes h) | e_{k-\theta(y),\tau} \otimes h) \\ &= (1 - e^{-\beta} d) e^{-\beta \theta(y)} \sum_{j=0}^{\infty} \sum_{\tau \in \Sigma_j} e^{-\beta j} (\pi(T_{b^t})(e_{j,\tau} \otimes h) | e_{j,\tau} \otimes h) \\ &= e^{-\beta \theta(y)} \psi_h(T_{b^t}) \\ &= e^{-\beta \theta(y)} \psi_h(T_{y^{-1}x}). \end{aligned}$$

Thus ψ_h satisfies (5.2), and Proposition 5.1 implies that ψ_h is a KMS_{β} state. \square

The subgroup K is a copy of the additive group \mathbb{Z} , written multiplicatively because it sits inside the nonabelian group G . Thus $C^*(K)$ is isomorphic to the algebra $C(\mathbb{T})$, and states on $C^*(K)$ are given by probability measures μ on \mathbb{T} . For such a measure μ , we consider the representation $W = W(\mu)$ of K on $H = L^2(\mathbb{T}, d\mu)$ given by

$$(6.9) \quad (W_{b^t} f)(z) = z^t f(z),$$

and the unit vector $h = 1_\mu$ in $L^2(\mathbb{T}, d\mu)$ associated to the constant function 1. Then Proposition 6.3 gives us a KMS_β state $\psi_{\beta, \mu} := \psi_{1_\mu}$, and we need to calculate the values of this state on the elements T_{b^t} , which by Corollary 5.2 determine the state. For $k = 0$, we have just the trivial stem e , and

$$(6.10) \quad (\pi(T_{b^t})(e_{k,e} \otimes 1_\mu) \mid e_{k,e} \otimes 1_\mu) = (W_{b^t} 1_\mu \mid 1_\mu) = \int_{\mathbb{T}} z^t d\mu(z).$$

For $k \geq 1$ and each stem $\sigma \in \Sigma_k$, we have

$$(6.11) \quad \pi(T_{b^t})(e_{k,\sigma} \otimes 1_\mu) = e_{k, \text{stem}(b^t \sigma)} \otimes W_{b^s} 1_\mu \quad \text{where } b^t \sigma = \text{stem}(b^t \sigma) b^s.$$

Thus

$$(\pi(T_{b^t})(e_{k,\sigma} \otimes 1_\mu) \mid e_{k,\sigma} \otimes 1_\mu)$$

vanishes unless $\text{stem}(b^t \sigma) = \sigma$, and hence we need to know when this happens.

Lemma 6.4. *Suppose that $\sigma \in \Sigma_k$ for some $k \geq 1$ and $b^t \in K$. Then $\text{stem}(b^t \sigma) = \sigma$ if and only if d divides $c^j d^{-j} t$ for every j such that $1 \leq j < k$. If so, we have*

$$(6.12) \quad b^t \sigma = \sigma b^{c^k d^{-k} t}.$$

Proof. Suppose first that $\text{stem}(b^t \sigma) = \sigma$, and write $\sigma = b^{t_0} a b^{t_1} a \cdots b^{t_{k-1}} a$ in normal form. Then since $\text{stem}(b^t \sigma)$ begins with $\text{stem}(b^t b^{t_0} a)$, we have $\text{stem}(b^t b^{t_0} a) = b^{t_0} a$. Write $t + t_0 = r + nd$ with $0 \leq r < d$, and then $b^r a = \text{stem}(b^t b^{t_0} a) = b^{t_0} a$. Now uniqueness of the normal form gives $r = t_0$, $t = nd$ and $b^t b^{t_0} a = b^{t_0} a b^{nc} = b^{t_0} a b^{cd^{-1}t}$. Now running a similar argument on $b^{cd^{-1}t} b^{t_1} a$ shows that d divides $cd^{-1}t$, and $b^{cd^{-1}t} b^{t_1} a = b^{t_1} a b^{c^2 d^{-2}t}$. Continuing this way shows that d divides $c^j d^{-j} t$ for every $j < k$, and gives the formula (6.12).

For the converse, just note that the condition on d allows us to pull b^t through σ without changing the powers b^{t_j} for $j \leq k$. \square

Remark 6.5. When c and d are coprime, d divides $c^j d^{-j} t$ if and only if d^{j+1} divides t , and hence left multiplication by b^t fixes all the stems in Σ_k if and only if d^k divides t . However, in general it is possible that d divides $c^j d^{-j} t$ but does not divide $c^{j-1} d^{-(j-1)} t$. For example, consider $c = 8$ and $d = 12$. Then $cd^{-1}t = 2t/3$ and $c^2 d^{-2}t = 4t/9$. Thus with $t = 27$, $d = 12$ divides $c^2 d^{-2}t$ but not $cd^{-1}t$.

Equation (6.10) tells us where the first integral in (6.1) comes from. For $k \geq 1$, Lemma 6.4 tells us why only the summands described in (6.1) survive. Suppose that $k \in \mathbb{N}$ and d divides $c^j d^{-j} t$ whenever $1 \leq j < k$. Then (6.12) tells us that the s in (6.11)

is $c^k d^{-k} t$. Thus the k th summand in the formula (6.6) for $\psi_{\beta,\mu}(T_{b^t}) = \phi_{1_\mu}(T_{b^t})$ is

$$\begin{aligned}
& (1 - e^{-\beta} d) \sum_{\sigma \in \Sigma_k} e^{-\beta k} (\pi(T_{b^t})(e_{k,\sigma} \otimes 1_\mu) \mid e_{k,\sigma} \otimes 1_\mu) \\
&= (1 - e^{-\beta} d) \sum_{\sigma \in \Sigma_k} e^{-\beta k} (e_{k,\sigma} \otimes W_{b^{c^k d^{-k} t}} 1_\mu \mid e_{k,\sigma} \otimes 1_\mu) \\
&= (1 - e^{-\beta} d) \sum_{\sigma \in \Sigma_k} e^{-\beta k} \int_{\mathbb{T}} z^{c^k d^{-k} t} d\mu(z) \\
&= (1 - e^{-\beta} d) |\Sigma_k| e^{-\beta k} \int_{\mathbb{T}} z^{c^k d^{-k} t} d\mu(z) \\
&= (1 - e^{-\beta} d) d^k e^{-\beta k} \int_{\mathbb{T}} z^{c^k d^{-k} t} d\mu(z).
\end{aligned}$$

Thus we recover the formula (6.1) for $\psi_{\beta,\mu}(T_{b^t})$.

Next we have to prove that every KMS_β state has the form $\psi_{\beta,\mu}$. So we suppose that ϕ is a KMS_β state. Recall that the set

$$\{T_x : x \in \Sigma_1\} = \{U^i V : 0 \leq i < d\}$$

is a Toeplitz-Cuntz-Krieger family, and set

$$P := 1 - \sum_{x \in \Sigma_1} T_x T_x^*.$$

Then the KMS condition implies that $\phi(P) = 1 - e^{-\beta} d$, so we may consider the *conditioned state*

$$\phi_P : a \mapsto (1 - e^{-\beta} d)^{-1} \phi(P a P).$$

This is in particular a state on the C^* -subalgebra $C^*(K) = C^*(T_b) \cong C(\mathbb{T})$, and hence is given by a measure μ on \mathbb{T} : we choose the measure that satisfies

$$\phi_P(T_{b^n}) = \int_{\mathbb{T}} z^n d\mu(z) \quad \text{for } n \in \mathbb{N}.$$

We aim to prove that $\phi = \psi_{\beta,\mu}$. The argument follows closely that of [12, Proposition 7.1], though the details in the calculations are quite different.

We begin by claiming that

$$\{T_x P T_x^* : x \in \bigcup_{k=0}^{\infty} \Sigma_k\}$$

is a family of mutually orthogonal projections. To see this, take $x \in \Sigma_j$ and $y \in \Sigma_k$ and consider $P T_x^* T_y P$. If $j = k$, then either $x = y$ or $x \vee y = \infty$, and Nica covariance gives $P T_x^* T_y P = \delta_{x,y} P$. If $j \neq k$, then Nica covariance leaves a factor of the form $P T_\sigma$ or $T_\sigma^* P$ with $\theta(\sigma) > 0$. Now we write $\sigma = w \sigma'$ with $w \in \Sigma_1$, and then

$$\begin{aligned}
P T_\sigma &= \left(1 - \sum_{z \in \Sigma_1} T_z T_z^*\right) T_w T_{\sigma'} \\
&= \left(T_w - \sum_{z \in \Sigma_1} T_z T_z^* T_w\right) T_{\sigma'} \\
&= (T_w - T_w) T_{\sigma'} = 0.
\end{aligned}$$

Thus $P T_x^* T_y P = 0$ when $j \neq k$, and the claim follows.

Now for each n ,

$$P_n := \sum_{k=0}^n \sum_{x \in \Sigma_k} T_x P T_x^*$$

is a projection. Then as in the proof of [12, Proposition 7.2], the KMS condition implies that

$$\begin{aligned} \phi(P_n) &= \sum_{k=0}^n \sum_{x \in \Sigma_k} \phi(T_x P T_x^*) = \sum_{k=0}^n \sum_{x \in \Sigma_k} e^{-\beta k} \phi(P T_x^* T_x) \\ &= \phi(P) \sum_{k=0}^n e^{-\beta k} d^k = (1 - e^{-\beta} d) \sum_{k=0}^n e^{-\beta k} d^k \end{aligned}$$

converges to 1 as $n \rightarrow \infty$. It follows from [12, Lemma 7.3] that for each $c \in C^*(G, P)$, we have $\phi(P_n c P_n) \rightarrow \phi(c)$ as $n \rightarrow \infty$. We now use the KMS condition to simplify

$$\begin{aligned} \phi(c) &= \lim_{n \rightarrow \infty} \phi(P_n c P_n) = \lim_{n \rightarrow \infty} \sum_{j,k=0}^n \sum_{x \in \Sigma_j, y \in \Sigma_k} \phi((T_x P T_x^*) c (T_y P T_y^*)) \\ &= \lim_{n \rightarrow \infty} \sum_{j,k=0}^n \sum_{x \in \Sigma_j, y \in \Sigma_k} e^{-\beta j} \phi(P T_x^* c T_y P T_y^* T_x P) \\ &= \lim_{n \rightarrow \infty} \sum_{k=0}^n \sum_{x \in \Sigma_k} e^{-\beta k} \phi(P T_x^* c T_x P) \\ &= \lim_{n \rightarrow \infty} (1 - e^{-\beta} d) \sum_{k=0}^n \sum_{x \in \Sigma_k} e^{-\beta k} \phi_P(T_x^* c T_x). \end{aligned}$$

This is an analogue of the reconstruction formula of [12, Proposition 7.2].

To finish off, we take $c = T_{b^t}$ in the reconstruction formula. Then for $x \in \Sigma_k$, $T_x^* T_{b^t} T_x$ has the form $T_x^* T_{\text{stem}(b^t x)} T_{b^s}$, which vanishes unless $\text{stem}(b^t x) = x$. In that case Lemma 6.4 implies that d divides $c^j d^{-j} t$ for all $j < k$, $b^t x = x b^{c^k d^{-k} t}$ and $T_x^* T_{\text{stem}(b^t x)} T_{b^{c^k d^{-k} t}} = T_{b^{c^k d^{-k} t}}$. (We have to worry separately about $k = 0$, though.) Thus we have

$$\phi(T_{b^t}) = (1 - e^{-\beta} d) \left(\phi_P(T_{b^t}) + \sum_{\{k \geq 1 : d \mid c^j d^{-j} t \text{ for } 0 \leq j < k\}} e^{-\beta k} \phi_P(T_{b^{c^k d^{-k} t}}) \right),$$

which by definition of the measure μ is the same as $\psi_{\beta, \mu}(T_{b^t})$. Thus Corollary 5.2 implies that $\phi = \phi_{\beta, \mu}$.

Now we add the assumption that d does not divide c , and aim to prove that $\mu \mapsto \psi_{\beta, \mu}$ is injective. So we suppose that μ and ν are probability measures on \mathbb{T} satisfying $\psi_{\beta, \mu} = \psi_{\beta, \nu}$. To simplify the notation, we observe that the integrals appearing in our formulas are the moments

$$M_n(\mu) := \int_{\mathbb{T}} z^n d\mu$$

of the measures. Since the non-negative moments characterise a probability measure, we will prove that $M_t(\mu) = M_t(\nu)$ for all $t \in \mathbb{N}$.

We prove by induction on k that, if $t \in \mathbb{N}$, d divides $c^j d^{-j} t$ for all $j < k$ and d does not divide $c^k d^{-k} t$, then

$$(6.13) \quad M_{c^j d^{-j} t}(\mu) = M_{c^j d^{-j} t}(\nu) \quad \text{for all } j \text{ satisfying } 0 \leq j \leq k.$$

For $k = 0$, which we interpret to mean that d does not divide t , the sum in (6.1) is absent, and we have

$$(1 - e^{-\beta} d) M_t(\mu) = \psi_{\beta, \mu}(T_{b^t}) = \psi_{\beta, \nu}(T_{b^t}) = (1 - e^{-\beta} d) M_t(\nu).$$

Since $e^\beta > d$, we deduce that $M_t(\mu) = M_t(\nu)$.

Suppose that the inductive hypothesis is true for k , and we have $s \in \mathbb{N}$ such that d divides $c^j d^{-j} s$ for $j < k + 1$ and d does not divide $c^{k+1} d^{-(k+1)} s$. Then (6.1) gives

$$\psi_{\beta, \mu}(T_{b^s}) = (1 - e^{-\beta} d) (M_s(\mu) + e^{-\beta} d M_{cd^{-1}s}(\mu) + \cdots + e^{-\beta(k+1)} d^{k+1} M_{c^{k+1} d^{-(k+1)} s}(\mu)),$$

and hence $\psi_{\beta, \mu} = \psi_{\beta, \nu}$ implies that

$$(6.14) \quad \begin{aligned} M_s(\mu) + e^{-\beta} d M_{cd^{-1}s}(\mu) + \cdots + e^{-\beta(k+1)} d^{k+1} M_{c^{k+1} d^{-(k+1)} s}(\mu) \\ = M_s(\nu) + e^{-\beta} d M_{cd^{-1}s}(\nu) + \cdots + e^{-\beta(k+1)} d^{k+1} M_{c^{k+1} d^{-(k+1)} s}(\nu). \end{aligned}$$

Notice that $t = cd^{-1}s$ has the property that d divides $c^j d^{-j} t$ for all $j < k$ and d does not divide $c^k d^{-k} t$. Thus the inductive hypothesis implies that

$$M_{c^{j+1} d^{-(j+1)} s}(\mu) = M_{c^j d^{-j} t}(\mu) = M_{c^j d^{-j} t}(\nu) = M_{c^{j+1} d^{-(j+1)} s}(\nu) \quad \text{for } 0 \leq j < k.$$

But this says precisely that (6.13) holds for $1 \leq j \leq k + 1$. Now cancelling terms in (6.14) gives $M_s(\mu) = M_s(\nu)$, which is (6.13) for the remaining case $j = 0$. Thus we have proved the inductive hypothesis for $k + 1$, and this completes the proof by induction.

Now we fix $t \in \mathbb{N}$. Since d does not divide c , there is a first k such that d does not divide $c^k d^{-k} t$, and then we have (6.13) for this k . But now taking $j = 0$ in (6.13) shows that $M_t(\mu) = M_t(\nu)$. Thus μ and ν have the same moments, and $\mu = \nu$. Thus $\mu \mapsto \psi_{\beta, \mu}$ is injective.

Since the sum in (6.1) is always finite, the assignment $\mu \mapsto \psi_{\beta, \mu}$ is affine and continuous for the weak* topologies. We have shown that it is a bijection of the compact space $P(\mathbb{T})$ onto the simplex of KMS_β states. Hence it is a homeomorphism, and this completes the proof of Theorem 6.1.

7. KMS STATES AT THE CRITICAL INVERSE TEMPERATURE

Recall from Corollary 5.3 that every $\text{KMS}_{\ln d}$ state of $C^*(G, P)$ factors through the quotient map of $C^*(G, P)$ onto the Cuntz algebra $\mathcal{O}(G, P)$. We write \bar{T}_x for the image of $T_x \in C^*(G, P)$ in $\mathcal{O}(G, P)$.

Proposition 7.1. *There is a $\text{KMS}_{\ln d}$ state ψ on $(\mathcal{O}(G, P), \alpha)$ such that*

$$\psi(\bar{T}_x \bar{T}_y^*) = \delta_{x,y} e^{-\beta \theta(x)}.$$

If d does not divide c , then this is the only KMS state on $(\mathcal{O}(G, P), \alpha)$.

Lemma 7.2. *Suppose that d does not divide c and ϕ is a $\text{KMS}_{\ln d}$ state of the Cuntz system $(\mathcal{O}(G, P), \alpha)$. Then $\phi(\bar{T}_{b^t}) = 0$ for all $t \neq 0$.*

Proof. Suppose that $\phi(\bar{T}_{b^t}) \neq 0$. Since $\{S_j := \bar{T}_{b^j a} : 0 \leq j < d\}$ is a Cuntz family, so is

$$\{S_\mu := S_{\mu_1} S_{\mu_2} \cdots S_{\mu_k} : 0 \leq \mu_i < d\} = \{\bar{T}_\sigma : \sigma \in \Sigma_k\}.$$

Thus for every k ,

$$\phi(\bar{T}_{b^t}) = \phi\left(\bar{T}_{b^t} \sum_{\sigma \in \Sigma_k} \bar{T}_\sigma \bar{T}_\sigma^*\right) = \sum_{\sigma \in \Sigma_k} \phi(\bar{T}_{b^t \sigma} \bar{T}_\sigma^*)$$

and there exists $\sigma \in \Sigma_k$ such that $\phi(\bar{T}_{b^t \sigma} \bar{T}_\sigma^*) \neq 0$. Proposition 5.1 then implies that $b^t \sigma \vee \sigma < \infty$, and since $b^t \sigma$ and σ have the same height, we must have $\text{stem}(b^t \sigma) = \sigma$. Thus we can apply Lemma 6.4 for every k , and deduce that d divides $c^j d^{-j} t$ for all $j \geq 1$. Write $g := \gcd(c, d)$, and define $c_1 := g^{-1}c$ and $d_1 := g^{-1}d$. Then $\gcd(c_1, d_1) = 1$ and for all j we have

$$\begin{aligned} d \text{ divides } c^j d^{-j} t &\implies d_1 \text{ divides } c_1^j d_1^{-j} t \\ &\implies d_1 \text{ divides } d_1^{-j} t \\ &\implies d_1^{j+1} \text{ divides } t. \end{aligned}$$

Since d does not divide c , we have $d_1 > 1$, and hence $t = 0$. \square

Proof of Proposition 7.1. We choose a decreasing sequence $\{\beta_n\}$ such that $\beta_n \rightarrow \ln d$, and take μ to be the Haar measure on \mathbb{T} . Then by passing to a subsequence, we may assume that $\{\psi_{\beta_n, \mu}\}$ converges weak* to a state ϕ of $(C^*(G, P), \alpha)$. Then it follows from [1, Proposition 5.3.23] that ϕ is a $\text{KMS}_{\ln d}$ state of $(C^*(G, P), \alpha)$. Corollary 5.3 implies that ϕ factors through a state ψ of $(\mathcal{O}(G, P), \alpha)$. Since the non-zero moments of the Haar measure all vanish, we have $\psi(\bar{T}_{b^t}) = 0$ for all $t \neq 0$, and then the formula for ψ follows.

For uniqueness, we use Lemma 7.2 to see that any other $\text{KMS}_{\ln d}$ state agrees with ψ on the elements \bar{T}_{b^t} , and hence by Corollary 5.2 on all of $\mathcal{O}(G, P)$. \square

Example 7.3. Suppose that d divides c , that $\beta > \ln d$, and that $\mu \in P(\mathbb{T})$. Then either d divides $c^j d^{-j} t$ for all $j \geq 0$, or d does not divide t . Thus

$$\psi_{\beta, \mu}(T_{b^t}) = \begin{cases} (1 - e^{-\beta} d)(M_t(\mu) + \sum_{k=1}^{\infty} e^{-\beta k} d^k M_{c^k d^{-k} t}(\mu)) & \text{if } d \text{ divides } t \\ (1 - e^{-\beta} d)M_t(\mu) & \text{if } d \text{ does not divide } t. \end{cases}$$

If $d = c$, then the only moment appearing in our formula is $M_t(\mu)$, and summing the geometric series shows that

$$\psi_{\beta, \mu}(T_{b^t}) = \begin{cases} M_t(\mu) & \text{if } d \text{ divides } t \\ (1 - e^{-\beta} d)M_t(\mu) & \text{otherwise.} \end{cases}$$

Now the procedure in the proof of Proposition 7.1 gives a $\text{KMS}_{\ln d}$ state on $(\mathcal{O}(G, P), \alpha)$, and measures with different moments M_{dn} will give different states.

If $d \neq c$, then we write δ_z for the point mass at $z \in \mathbb{T}$, and take

$$\mu = \frac{1}{cd^{-1}} \sum_{w^{cd^{-1}}=1} \delta_w.$$

Then μ is a probability measure with moments

$$M_n(\mu) = \begin{cases} 1 & \text{if } cd^{-1} \text{ divides } n \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\psi_{\beta,\mu}(T_{bt}) = \begin{cases} 1 & \text{if } cd^{-1} \text{ and } d \text{ divide } t \\ (1 - e^{-\beta}d) & \text{if } cd^{-1} \text{ divides } t \text{ and } d \text{ does not} \\ 0 & \text{if } cd^{-1} \text{ does not divide } t. \end{cases}$$

If we now choose β_n decreasing to $\ln d$ as in the proof of Proposition 7.1, we get a $\text{KMS}_{\ln d}$ state ψ_μ on $(\mathcal{O}(G, P), \alpha)$ such that

$$\psi_\mu(\bar{T}_{bt}) = \begin{cases} 1 & \text{if } cd^{-1} \text{ and } d \text{ divide } t \\ 0 & \text{otherwise.} \end{cases}$$

So uniqueness of the $\text{KMS}_{\ln d}$ state fails whenever d divides c .

Example 7.4. The case $d = c$ is quite different. Then the unitary element \bar{T}_{b^c} commutes with everything. It and the Cuntz family $\{\bar{T}_{b^j a} : 0 \leq j < c\}$ generate $\mathcal{O}(G, P)$: indeed, \bar{T}_a is a member of the Cuntz family, and we can recover \bar{T}_b using the formula

$$\sum_{j=0}^{c-2} \bar{T}_{b^{j+1}a} \bar{T}_{b^j a}^* + \bar{T}_{b^c} \bar{T}_a \bar{T}_{b^{c-1}a} = \bar{T}_b \left(\sum_{j=0}^{c-1} \bar{T}_{b^j a} \bar{T}_{b^j a}^* \right) = \bar{T}_b.$$

The Cuntz family gives us a homomorphism $\pi : \mathcal{O}_c = C^*(S_j) \rightarrow \mathcal{O}(G, P)$ such that $\pi(S_j) = \bar{T}_{b^j a}$, and the unitary \bar{T}_{b^c} gives us a homomorphism $\rho : C(\mathbb{T}) \rightarrow \mathcal{O}(G, P)$ such that $\rho(\iota) = \bar{T}_{b^c}$. Since they have commuting ranges, they induce a homomorphism $\pi \otimes \rho$ of $\mathcal{O}_c \otimes C(\mathbb{T})$ onto $\mathcal{O}(G, P)$. This is in fact an isomorphism because the universal property of $\mathcal{O}(G, P)$ gives an inverse.

Remark 7.5. In the definition of the Baumslag-Solitar group $\text{BS}(c, d)$, the integers c and d play an equal role. So at first sight it seems strange that the critical inverse temperature is dictated by d alone. However, right at the beginning of his theory, Nica made a critical choice: his covariance relation was modelled on the behaviour of the *left*-regular representation. If he had started with the right-regular representation, his theory would have looked quite different.

The right-regular representation ρ of a group G is characterised in terms of the usual basis $\{e_g : g \in G\}$ for $\ell^2(G)$ by $\rho_h e_g = e_{gh^{-1}}$ (the inverse has to be there to ensure that $\rho_g \rho_h = \rho_{gh}$). So the natural representation of P by operators $\{R_x : x \in P\}$ on $\ell^2(P)$ is given by

$$R_x e_y = \begin{cases} 0 & \text{if } y \notin Px \\ e_{yx^{-1}} & \text{if } y \in Px. \end{cases}$$

Each R_x is a *coisometry*: R_x^* is an isometry. In other words, R_x is a partial isometry with initial projection $R_x^* R_x : \ell^2(P) \rightarrow \ell^2(Px)$ and range projection $R_x R_x^* = 1$.

The analogue of Nica's partial order is the right-invariant order defined by $x \leq_r y \iff y \in Px$ (and for the duration of this remark, we'll write \leq_l for the usual left-invariant one). There is an analogous notion of “right-quasi-lattice ordered” involving least upper bounds $x \vee_r y$ with respect to \leq_r , and the analogue of Nica covariance is the relation

$$(7.1) \quad (R_x^* R_x)(R_y^* R_y) = \begin{cases} R_{x \vee_r y}^* R_{x \vee_r y} & \text{if } x \vee_r y < \infty \\ 0 & \text{if } x \vee_r y = \infty, \end{cases}$$

which is satisfied by the right-regular representation. One can then get a universal C^* -algebra, which we will denote by $C^*(G, P, \leq_r)$.

Fortunately, there is a device for studying this C^* -algebra (for which we thank Ilija Tolich). Consider the opposite group $G^{\text{op}} = \{g^b : g \in G\}$ with $g^b h^b = (hg)^b$, and the corresponding subsemigroup P^{op} . Then the usual partial order \leq_1 on G^{op} satisfies

$$g^b \leq_1 h^b \iff h^b \in g^b P^{\text{op}} = (Pg)^b \iff h \in Pg \iff g \leq_r h;$$

we deduce that $(G^{\text{op}}, P^{\text{op}})$ is quasi-lattice ordered in the usual sense if and only if (G, P, \leq_r) is right-quasi-lattice ordered, with $g^b \vee_1 h^b = (g \vee_r h)^b$. One can check quite easily that $R^T : x \mapsto T_x^*$ is a covariant coisometric representation of P in the sense that (7.1) holds if and only if T is a Nica-covariant representation of P^{op} . Thus $(C^*(G^{\text{op}}, P^{\text{op}}), R^T)$ is universal for coisometric representations satisfying (7.1).

When $G = \text{BS}(c, d)$, the opposite group is

$$G^{\text{op}} = \langle a, b : b^c a = ab^d \rangle = \text{BS}(d, c).$$

Thus, had we chosen to work with the partial order \leq_r , we would have found a system with a phase transition at inverse temperature $\ln c$. (There is a minor wrinkle: because passing from the isometric representation T to the coisometric representation R^T involves an adjoint, the dynamics satisfies $\alpha_t^r(R_x) = e^{-i\theta(x)} R_x$. However, one can argue that this is the natural one, because both this and the usual dynamics are implemented spatially on $\ell^2(P)$ by the unitary representation U such that $U_t e_x = e^{it\theta(x)} e_x$.)

8. GROUND STATES

A state ϕ is a *ground state* of $(C^*(G, P), \alpha)$ if, for all analytic elements a and b , the entire function $z \mapsto \phi(a\alpha_z(b))$ is bounded in the upper half-plane $\text{Im} z > 0$. The KMS_∞ states are the weak* limits of sequences of KMS_{β_n} states as $\beta_n \rightarrow \infty$. Every KMS_∞ state is a ground state, but a ground state need not be a KMS_∞ state by [1, Proposition 5.3.23] and [3, Proposition 3.8].

Theorem 8.1. *Suppose that ω is a state of the Toeplitz algebra $\mathcal{T}(\mathbb{N}) = C^*(S)$. Then there is a ground state ψ_ω of $(C^*(G, P), \alpha)$ such that*

$$(8.1) \quad \psi_\omega(T_x T_y^*) = \begin{cases} 0 & \text{if } \theta(x) \neq 0 \text{ or } \theta(y) \neq 0 \\ \omega(S^s S^{*t}) & \text{if } x = b^s \text{ and } y = b^t. \end{cases}$$

The state ψ_ω is a KMS_∞ state if and only if ω factors through the quotient map $q : \mathcal{T}(\mathbb{N}) \rightarrow C(\mathbb{T})$. The map $\omega \mapsto \psi_\omega$ is an affine isomorphism of the state space of $\mathcal{T}(\mathbb{N})$ onto the compact convex set of ground states.

There are many states of $\mathcal{T}(\mathbb{N})$ which do not factor through $q : \mathcal{T}(\mathbb{N}) \rightarrow C(\mathbb{T})$: for example, the vector states given by unit vectors in ℓ^2 . Thus Theorem 8.1 implies that the system $(C^*(G, P), \alpha)$ has many ground states which are not KMS_∞ states. Thus (in the terminology of [3]) the system admits a second phase transition at $\beta = \infty$.

We now do some preparation for the the proof of Theorem 8.1. First we need to be able to recognise ground states.

Lemma 8.2. *A state ψ of $(C^*(G, P), \alpha)$ is a ground state if and only if*

$$(8.2) \quad \psi(T_x T_y^*) \neq 0 \implies \theta(x) = \theta(y) = 0.$$

Proof. Suppose that ψ is a ground state and $\psi(T_x T_y^*) \neq 0$. Then

$$|\psi(T_x \alpha_{r+is}(T_y^*))| = |e^{-i(r+is)\theta(y)} \psi(T_x T_y^*)| = e^{s\theta(y)} |\psi(T_x T_y^*)|$$

is bounded on the upper half-plane $s \geq 0$, and hence $\theta(y) = 0$. Since $\psi(T_y T_x^*) = \overline{\psi(T_x T_y^*)} \neq 0$, we also deduce that $\theta(x) = 0$.

Next suppose that ψ is a state satisfying (8.2). Let $X = T_x T_y^*$ and Y be analytic elements for α . Then the Cauchy-Schwarz inequality gives

$$\begin{aligned} (8.3) \quad |\psi(Y \alpha_{r+is}(X))|^2 &= e^{-2s(\theta(x)-\theta(y))} |\psi(Y T_x T_y^*)|^2 \\ &\leq e^{-2s(\theta(x)-\theta(y))} \psi(Y^* Y) \psi(T_y T_x^* T_x T_y^*) \\ &= e^{-2s(\theta(x)-\theta(y))} \psi(Y^* Y) \psi(T_y T_y^*). \end{aligned}$$

If $\theta(y) \neq 0$ then (8.2) implies that $\psi(T_y T_y^*) = 0$, and the right-hand side of (8.3) is trivially bounded. If $\theta(y) = 0$, then the right-hand side of (8.3) is bounded by $\psi(Y^* Y) \psi(T_y T_y^*)$. So either way, $|\psi(Y \alpha_{r+is}(X))|$ is bounded for $s \geq 0$, and ψ is a ground state. \square

Next we need a good supply of representations. Our basic construction was inspired by our earlier one using induced representations.

We continue to use the orthonormal basis $\{e_{k,\sigma} : \sigma \in \Sigma_k\}$ for $\ell^2(\Sigma_k)$.

Lemma 8.3. *Suppose that W is an isometry of a Hilbert space H . Then there are isometries U and V on $\bigoplus_{k=0} \ell^2(\Sigma_k) \otimes H$ such that*

$$\begin{aligned} U(e_{k,\sigma} \otimes h) &= e_{k,\text{stem}(b\sigma)} \otimes W^s h \quad \text{where } b\sigma = \text{stem}(b\sigma)b^s, \text{ and} \\ V(e_{k,\sigma} \otimes h) &= e_{k+1,a\sigma} \otimes h. \end{aligned}$$

Proof. Lemma 3.2 implies that $\sigma \mapsto \text{stem}(b\sigma)$ is a bijection of Σ_k onto Σ_k . Thus if $\{h_i : i \in I\}$ is an orthonormal basis for H , then $\{e_{k,\text{stem}(b\sigma)} \otimes W^s h_i : \sigma \in \Sigma, i \in I\}$ is an orthonormal set in $\ell^2(\Sigma_k) \otimes H$ for each k . Thus there is an isometry U as claimed. Since each $a\sigma$ is already a stem, Lemma 3.2 also implies that $\sigma \mapsto a\sigma$ is an injection of Σ_k in Σ_{k+1} for each k . Thus $\{e_{k+1,a\sigma} \otimes h_i\}$ is also orthonormal, and there is an isometry V with the required property. \square

Proposition 8.4. *Suppose that W is an isometry of a Hilbert space H , and U, V are as in Lemma 8.3. Then U and V satisfy the relations (a), (b) and (c) of Proposition 4.1.*

Proof. The calculations in the fourth paragraph of the proof of Proposition 6.2 show that U and V satisfy (a) and (c). To verify (b), we need a formula for U^* . We claim that

$$(8.4) \quad U^*(e_{k,\tau} \otimes h) = e_{k,\rho} \otimes W^{*t} h \quad \text{where } \rho \in \Sigma_k \text{ satisfies } \tau b^t = b\rho;$$

Lemma 3.2 implies that there is a unique stem ρ such that $b\rho$ begins with τ , and then t is uniquely determined by $\tau b^t = b\rho$. To prove the claim, we compare

$$(8.5) \quad (e_{k,\rho} \otimes W^{*t} h | e_{k,\sigma} \otimes g) = \delta_{\rho,\sigma} (h | W^t g) \quad \text{where } \tau b^t = b\rho$$

with

$$(8.6) \quad (e_{k,\tau} \otimes h | U(e_{k,\sigma} \otimes g)) = \delta_{\tau,\text{stem}(b\sigma)} (h | W^s g) \quad \text{where } b\sigma = \text{stem}(b\sigma)b^s.$$

First, suppose that $\rho = \sigma$. Then $b\sigma = \text{stem}(b\sigma)b^s = \text{stem}(b\rho)b^s = \text{stem}(\tau b^t)b^s = \tau b^s$ because τ is a stem. Thus $\tau = \text{stem}(b\sigma)$. Now $\tau b^t = b\rho = b\sigma = \text{stem}(b\sigma)b^s = \tau b^s$, and hence $s = t$. Thus (8.5) and (8.6) agree. Second, suppose that $\rho \neq \sigma$. By Lemma 3.2 (a),

$x \mapsto \text{stem}(bx)$ is a bijection on Σ_k , and hence $\text{stem}(b\sigma) \neq \text{stem}(b\rho) = \text{stem}(\tau b^t) = \tau$, and both (8.5) and (8.6) are 0. This proves the claim.

We now compute the right-hand side of (b):

$$\begin{aligned} U^{d-1} V U^{*c}(e_{k,\sigma} \otimes h) &= U^{d-1} V(e_{k,\mu} \otimes W^{*t} h) \quad \text{where } \mu \in \Sigma_k \text{ satisfies } \sigma b^t = b^c \mu \\ &= e_{k+1, \text{stem}(b^{d-1} a \mu)} \otimes W^s W^{*t} h \quad \text{where } b^{d-1} a \mu = \text{stem}(b^{d-1} a \mu) b^s \\ &= e_{k+1, b^{d-1} a \mu} \otimes W^{*t} h \end{aligned}$$

because $b^{d-1} a \mu$ is a stem. The left-hand side of (b) is

$$U^* V(e_{k,\sigma} \otimes h) = e_{k+1, \rho} \otimes W^{*r} h \quad \text{where } \rho \in \Sigma_{k+1} \text{ satisfies } a \sigma b^r = b \rho.$$

Now the equation

$$b(b^{d-1} a \mu) = b^d a \mu = a b^c \mu = a \sigma b^t$$

implies that $\rho = b^{d-1} a \mu$ (because ρ is the unique stem such that $b\rho$ begins with $a\sigma$) and then $r = t$. Thus (b) follows. \square

Corollary 8.5. *Suppose that W is an isometry on a Hilbert space H , and U, V are the isometries described in Lemma 8.3. Let $\pi_{U,V}$ be the corresponding representation of $C^*(G, P)$ on $\bigoplus_{k \geq 0} \ell^2(\Sigma_k) \otimes H$. Then for every unit vector h in H , there is a ground state $\psi_{h,W}$ of $(C^*(G, P), \alpha)$ such that*

$$\psi_{h,W}(a) = (\pi_{U,V}(a)(e_{0,e} \otimes h) \mid e_{0,e} \otimes h).$$

Proof. Since T_x maps $\ell^2(\Sigma_0) \otimes H$ into $\ell^2(\Sigma_{\theta(x)}) \otimes H$,

$$\psi_{h,W}(T_x T_y^*) = (\pi_{U,V}(T_y)^*(e_{0,e} \otimes h) \mid \pi_{U,V}(T_x)^*(e_{0,e} \otimes h))$$

vanishes unless $\theta(x) = 0 = \theta(y)$. So Lemma 8.2 implies that $\psi_{h,W}$ is a ground state. \square

Proof of Theorem 8.1. Suppose ω is a state of $\mathcal{T}(\mathbb{N}) = C^*(S)$. We consider the GNS representation π_ω of $\mathcal{T}(\mathbb{N})$ on H_ω with cyclic vector ξ_ω , from which we can recover ω via the formula $\omega(c) = (\pi_\omega(c)\xi_\omega \mid \xi_\omega)$. Applying Corollary 8.5 with $W = \pi_\omega(S)$, U and V the isometries of Lemma 8.3, and $h = \xi_\omega$ gives a ground state $\psi_\omega := \psi_{\xi_\omega, \pi_\omega(S)}$ of $(C^*(G, P), \alpha)$ such that

$$\psi_\omega(a) = (\pi_{U,V}(a)(e_{0,e} \otimes \xi_\omega) \mid e_{0,e} \otimes \xi_\omega).$$

We need to verify the formula (8.1).

Since $V = \pi_{U,V}(T_a)$ maps $\ell^2(\Sigma_0) \otimes H_\omega = \mathbb{C}e_{0,e} \otimes H_\omega$ into $\ell^2(\Sigma_1) \otimes H_\omega$, we have

$$\psi_\omega(T_x T_y^*) = 0 \quad \text{unless } \theta(x) = 0 = \theta(y).$$

If $\theta(x) = 0 = \theta(y)$, then $x = b^s$ and $y = b^t$ for some $s, t \in \mathbb{N}$, and

$$\begin{aligned} \psi_\omega(T_x T_y^*) &= \psi_\omega(T_b^s T_b^{*t}) \\ &= (\pi_{U,V}(T_b^s T_b^{*t})(e_{0,e} \otimes \xi_\omega) \mid e_{0,e} \otimes \xi_\omega) \\ &= (U^s U^{*t}(e_{0,e} \otimes \xi_\omega) \mid e_{0,e} \otimes \xi_\omega) \\ &= (U^{*t}(e_{0,e} \otimes \xi_\omega) \mid U^{*s}(e_{0,e} \otimes \xi_\omega)). \end{aligned}$$

Since $\Sigma_0 = \{e\}$, the formula (8.4) for U^* collapses to $U^*(e_{0,e} \otimes h) = e_{0,e} \otimes W^*h$, and we have

$$\begin{aligned}\psi_\omega(T_x T_y^*) &= (e_{0,e} \otimes \pi_\omega(S)^{*t} \xi_\omega \mid e_{0,e} \otimes \pi_\omega(S)^{*s} \xi_\omega) \\ &= (\pi_\omega(S)^{*t} \xi_\omega \mid \pi_\omega(S)^{*s} \xi_\omega) \\ &= (\pi_\omega(S^s S^{*t}) \xi_\omega \mid \xi_\omega) \\ &= \omega(S^s S^{*t}),\end{aligned}$$

as in (8.1).

Next we suppose that ψ_ω is a KMS_∞ state. Then there are an increasing sequence $\beta_n \rightarrow \infty$ and KMS_{β_n} states ϕ_n such that ϕ_n converges weak* to ψ_ω . Corollary 5.3 implies that each ϕ_n factors through the quotient by the ideal generated by $1 - T_b T_b^*$, and hence so does the limit ψ_ω . The kernel of q is spanned by the elements $S^m(1 - SS^*)S^{*n}$ (they are a family of matrix units spanning $\ker q = \mathcal{K}(\ell^2)$), and the formula (8.1) implies that

$$\omega(S^m(1 - SS^*)S^{*n}) = \psi_\omega(T_{b^m}(1 - T_b T_b^*)T_{b^n}^*) = 0.$$

Thus ω factors through q .

Conversely, suppose that ω factors through q . Then there is a probability measure μ on \mathbb{T} such that $\omega(c) = \int q(c) d\mu$ for all $c \in \mathcal{T}(\mathbb{N})$. Choose a sequence β_n with $\beta_n \rightarrow \infty$. Then for each n , the state $\psi_{\beta_n, \mu}$ is determined by Corollary 5.2 and the formula (6.1) for $\psi_{\beta_n, \mu}(T_{b^t})$ in Proposition 6.1. The sum on the right-hand side of (6.1) is finite, and for each k we have

$$e^{-\beta_n k} d^k \int_{\mathbb{T}} z^{c^k d^{-k} t} d\mu(z) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since we also have $1 - e^{-\beta_n} d \rightarrow 1$, we deduce that

$$\psi_{\beta_n, \mu}(T_{b^t}) \rightarrow \int z^t d\mu(z) = \int q(S^t) d\mu = \psi_\omega(T_{b^t}).$$

Thus ψ_ω is a KMS_∞ state.

The formula (8.1) shows that $\omega \mapsto \psi_\omega$ is affine, weak* continuous and one-to-one. To see that it is onto, suppose ϕ is a ground state. Since T_b is a non-unitary isometry, Coburn's theorem implies that there is an isomorphism π_{T_b} of $\mathcal{T}(\mathbb{N})$ into $C^*(G, P)$ such that $\pi_{T_b}(S) = T_b$, and then $\omega := \psi \circ \pi_{T_b}$ is a state of $\mathcal{T}(\mathbb{N})$. Lemma 8.2 implies that ϕ vanishes on all spanning elements except those of the form $T_{b^s} T_{b^t}^*$, and formula (8.1) shows that ϕ agrees with ψ_ω on all spanning elements. Thus $\phi = \psi_\omega$, and $\omega \mapsto \psi_\omega$ is onto. Now we can deduce that it is a homeomorphism of the compact state space of $\mathcal{T}(\mathbb{N})$ onto the compact set of ground states. \square

APPENDIX A. AMENABILITY OF (G, P)

In setting up our conventions, we implicitly assumed that (G, P) is amenable in the sense of Nica, and here we prove this. This result is not a surprise, since Spielberg proved that his groupoid model is amenable [19, Theorem 3.23], and the various notions of amenability are all meant to do the same thing. Nevertheless, it is fairly routine to see it directly. For the purposes of this appendix, it is helpful to distinguish between the Toeplitz representation T of P on $\ell^2(P)$ and the universal representation of P in $C^*(G, P)$, which we denote by i (following [10]).

Theorem A.1. *The quasi-lattice ordered group (G, P) is amenable.*

We follow the argument of [10, Proposition 4.2], using the height map $\theta : G \rightarrow \mathbb{Z}$ in the role of the map ϕ in that proposition (which was later described as a “controlled map” in [5, §4]). Unfortunately, that proposition does not apply as it stands, since θ does not have the property (i) required of controlled maps in the statement of [10, Proposition 4.2] — for example, we have $a \leq ab$, but $\theta(a) = \theta(ab)$. But the general idea works.

By [10, Corollary 3.3], there is a contraction $\Phi : C^*(G, P) \rightarrow \overline{\text{span}}\{i(x)i(x)^* : x \in P\}$ such that

$$\Phi(i(x)i(y)^*) = \begin{cases} i(x)i(x)^* & \text{if } x = y \\ 0 & \text{otherwise.} \end{cases}$$

By [10, Definition 3.4], (G, P) is amenable if Φ is faithful in the sense that $\Phi(R^*R) = 0$ implies $R = 0$. We consider the dual action $\hat{\theta} : \mathbb{T} \rightarrow \text{Aut } C^*(G, P)$ characterised by $\hat{\theta}_z(i(x)) = z^{\theta(x)}i(x)$. Our strategy is to analyse the structure of the fixed-point algebra $C^*(G, P)^\theta = \overline{\text{span}}\{i(x)i(y)^* : \theta(x) = \theta(y)\}$ for this action, and show that Φ factors through the conditional expectation Φ^θ of $C^*(G, P)$ onto $C^*(G, P)^\theta$ obtained by averaging over \mathbb{T} .

Lemma A.2. *For $k \geq 0$, the algebraic linear span*

$$B_k := \text{span}\{i(\sigma)Di(\tau)^* : \sigma, \tau \in \Sigma_k \text{ and } D \in C^*(i(b))\}$$

is a closed C^ -subalgebra of $C^*(G, P)^\theta$.*

Proof. Since $\{i(\sigma) : \sigma \in \Sigma_k\}$ is a Toeplitz-Cuntz family, $\{i(\sigma)i(\tau)^* : \sigma, \tau \in \Sigma_k\}$ is a set of matrix units in the C^* -algebra $\overline{B_k}$. This gives a homomorphism $\phi : M_{\Sigma_k}(\mathbb{C}) \rightarrow \overline{B_k}$ which maps the usual matrix units $\{E_{\sigma\tau} : \sigma, \tau \in \Sigma_k\}$ to $\{i(\sigma)i(\tau)^*\}$. There is also a unital homomorphism $\psi : C^*(i(b)) \rightarrow \overline{B_k}$ such that

$$\psi(D) = \sum_{\sigma \in \Sigma_k} i(\sigma)Di(\sigma)^*.$$

We have

$$\begin{aligned} \phi(E_{\sigma\tau})\psi(D) &= i(\sigma)i(\tau)^* \sum_{\mu} i(\mu)Di(\mu)^* = i(\sigma)i(\tau)^*i(\tau)Di(\tau)^* \\ &= i(\sigma)Di(\tau)^* = \psi(D)\phi(E_{\sigma\tau}). \end{aligned}$$

Each $A \in M_{\Sigma_k}(\mathbb{C})$ is a linear combination of the $E_{\sigma\tau}$, and hence $\psi(D)\phi(A) = \phi(A)\psi(D)$ for all $A \in M_{\Sigma_k}(\mathbb{C})$ and $D \in C^*(i(b))$.

Since the ranges of ϕ and ψ commute, the universal property of the maximal tensor product gives a homomorphism $\phi \otimes_{\max} \psi$ of $M_{\Sigma_k}(\mathbb{C}) \otimes C^*(i(b))$ into $\overline{B_k}$. We claim that the range of $\phi \otimes_{\max} \psi$ is B_k . Since $M_{\Sigma_k}(\mathbb{C}) \otimes C^*(i(b))$ is spanned by elements of the form $E_{\sigma\tau} \otimes D$ (with no closure, see for example [18, Theorem B.18]), the range of $\phi \otimes_{\max} \psi$ is spanned by $\phi(E_{\sigma\tau})\psi(D) = i(\sigma)Di(\tau)^*$ (no closure) and hence equal to B_k . Thus B_k is a closed C^* -subalgebra of $C^*(G, P)^\theta$. \square

Lemma A.3. *For $k \geq 0$, we have $B_k B_{k+1} = B_{k+1}$.*

Proof. Since $\{i(\sigma) : \sigma \in \Sigma_k\}$ is a Toeplitz-Cuntz family, we have $i(\sigma)^*i(\tau) = 0$ unless σ extends τ or vice-versa. So to see $B_k B_{k+1} \subset B_{k+1}$, it suffices to take $\sigma, \tau, \mu, \nu \in \Sigma_k$, $\mu', \nu' \in \Sigma_1$, $C, D \in C^*(i(b))$, and show that

$$(A.1) \quad i(\sigma)Di(\tau)^*i(\mu\mu')Ci(\nu\nu')^* = \delta_{\tau,\mu}i(\sigma)Di(\mu')Ci(\nu\nu')^*$$

is in $B_k B_{k+1}$. Suppose that $D = i(b)^s i(b)^{*t}$ for some $s, t \in \mathbb{N}$. If $\mu' = b^j a$ for some integer $j \in [0, d]$, then

$$(A.2) \quad Di(\mu') = i(b)^s i(b)^{*t} i(\mu') = i(b)^s i(b)^{*t} i(b)^j i(a).$$

Now if $j < t$, then (A.2) is equal to

$$i(b)^{s+j-t} i(a) = i(\text{stem}(b^{s+j-t} a)) i(b)^q$$

for some $q \in \mathbb{N}$. On the other hand, if $t \geq j$, then (A.2) is equal to

$$i(b)^s i(b)^{(t-j)} i(a) = i(b)^s i(b)^{(t-j)(d-1)} i(a) i(b)^{(t-j)c}$$

and we write $i(b)^{s+(t-j)(d-1)} i(a)$ as $i(\text{stem}(b^{s+(t-j)(d-1)} a)) i(b)^r$ for some $r \in \mathbb{N}$. Either way, (A.1) has the form $i(\sigma \text{stem}(b^n a)) C' i(\mu\mu')$ and is in B_{k+1} . Thus $B_k B_{k+1} \subset B_{k+1}$.

To see the reverse containment, let $\sigma, \tau \in \Sigma_{k+1}$ and write $\sigma = \sigma' \sigma''$ where $\sigma' \in \Sigma_k$ and $\sigma'' \in \Sigma_1$. For $i(\sigma)Di(\tau)^* \in B_{k+1}$ we have

$$i(\sigma)Di(\tau)^* = i(\sigma' \sigma'')Di(\tau)^* = i(\sigma')i(\sigma'')^* i(\sigma')i(\sigma'')Di(\tau)^* = (i(\sigma')i(\sigma'')^*) (i(\sigma)Di(\tau)^*),$$

which is in $B_k B_{k+1}$. This extends to arbitrary elements of B_{k+1} and hence $B_k \subset B_k B_{k+1}$. \square

Corollary A.4. For $k \geq 0$, $C_k := B_0 + \cdots + B_k$ is a C^* -subalgebra of the core $C^*(G, P)^\theta$ and

$$C^*(G, P)^\theta = \overline{\bigcup_{k=0}^{\infty} C_k}.$$

Proof. We prove that C_k is a C^* -subalgebra of $C^*(G, P)^\theta$ by induction on k . Notice that $C_0 = B_0 = C^*(i(b))$ is a C^* -subalgebra.

Suppose that C_k is a C^* -subalgebra of $C^*(G, P)^\theta$ for $k \geq 0$. Lemma A.3 implies that

$$C_k B_{k+1} = (B_0 + \cdots + B_k) B_{k+1} = B_0 B_1 \cdots B_k B_{k+1} + \cdots + B_k B_{k+1} = B_{k+1}.$$

It follows that B_{k+1} is an ideal in the C^* -algebra A generated by C_k and B_{k+1} . Since C_k is a subalgebra of A , [14, Theorem 3.1.7] implies that $C_k + B_{k+1} = C_{k+1}$ is a C^* -subalgebra of A and hence of $C^*(G, P)^\theta$.

Since $C^*(G, P)^\theta = \overline{\text{span}\{i(x)i(y)^* : \theta(x) = \theta(y)\}}$, and such $i(x)i(y)^* \in B_{\theta(x)} \subset C_{\theta(x)}$, it follows that $\bigcup C_k$ is dense in $C^*(G, P)^\theta$, and hence $C^*(G, P)^\theta = \overline{\bigcup_{k=0}^{\infty} C_k}$. \square

The Toeplitz representation T of P on $\ell^2(P)$ is Nica covariant, and hence induces a homomorphism π_T of $C^*(G, P)$ onto the Toeplitz algebra $\mathcal{T}(G, P) := C^*(T_x : x \in P)$ such that $\pi_T \circ i = T$. We write

$$H_k = \overline{\text{span}\{e_{\sigma b^n} : \sigma \in \Sigma_k, n \in \mathbb{N}\}} \subset \ell^2(P), \text{ and then } \ell^2(P) = \bigoplus_{k \geq 0} H_k.$$

Lemma A.5. For $k \geq 0$,

- (a) H_k is invariant for $\pi_T|_{B_k}$ and
- (b) $\pi_T|_{B_k}$ is faithful on H_k .

Proof. For item (a), take $i(\sigma)i(b)^li(b)^{*m}i(\tau)^* \in B_k$ and $e_{\mu b^n} \in H_k$ where $m, n \in \mathbb{N}$. Then

$$\pi_T(i(\sigma)i(b)^li(b)^{*m}i(\tau)^*)e_{\mu b^n} = \begin{cases} e_{\sigma b^{l+(n-m)}} & \text{if } \mu = \tau \text{ and } n \geq m \\ 0 & \text{otherwise} \end{cases}$$

is again in H_k .

For item (b), take $B = \sum_{\rho, \mu \in \Sigma_k} i(\rho)D_{\rho, \mu}i(\mu)^* \in B_k$ and suppose $\pi_T(B)|_{H_k} = 0$. Fix $\sigma, \tau \in \Sigma_k$. Then

$$T_\sigma^* \pi_T(B) T_\tau = \pi_T(i(\sigma)^*) \pi_T(B) \pi_T(i(\tau)) = \pi_T(D_{\sigma, \tau}).$$

Since T_τ is an injection from H_0 into H_k and $\pi_T(B)|_{H_k} = 0$, we have $\pi_T(B)T_\tau|_{H_0} = 0$. Thus $\pi_T(D_{\sigma, \tau})|_{H_0} = 0$. But the restriction $(\pi_T|_{C^*(i(b))})|_{H_0}$ is generated by a nonunitary isometry and hence is faithful by Coburn's theorem (see, for example, [14, Theorem 3.5.18]). Thus $D_{\sigma, \tau} = 0$. It follows that $B = 0$. \square

Lemma A.6. π_T is faithful on the core $C^*(G, P)^\theta$.

Proof. Since $C^*(G, P)^\theta = \overline{\bigcup_k C_k}$ by Lemma A.4, it suffices to show that π_T is faithful on each C_k . Suppose $\pi_T(R) = 0$ where $R \in C_k$. Then there exist $R_i \in B_i$ such that $R = R_0 + \cdots + R_k$, and then $\pi_T(R_0) + \cdots + \pi_T(R_k) = 0$.

For stems σ and τ , if $\theta(\sigma) < \theta(\tau)$ then $T_\tau^* e_{\sigma b^n} = 0$. It follows that $\pi_T(R_i)|_{H_j} = 0$ when $j < i$. Thus

$$0 = \pi_T(R_0)|_{H_0} + \cdots + \pi_T(R_k)|_{H_0} = \pi_T(R_0)|_{H_0},$$

and Lemma A.5 implies that $R_0 = 0$. Then an induction argument gives each $R_i = 0$. Thus $R = 0$, as required. \square

Proof of Theorem A.1. We use an argument similar to that of [10, Proposition 4.2]. Let Φ^θ be the conditional expectation of $C^*(G, P)$ onto $C^*(G, P)^\theta$ obtained by averaging over the action $\hat{\theta}$, and recall that Φ^θ is faithful. Let $\{E_z\}$ be the usual orthonormal basis for $\ell^\infty(P)$. The diagonal map $\Delta : B(\ell^2(P)) \rightarrow \ell^\infty(P)$ given by

$$\Delta(T) = \sum_{z \in P} E_z T E_z$$

is faithful, and $\Delta \circ \pi_T = \pi_T \circ \Phi$ (see the computation on Page 433 of [10]). Now

$$\begin{aligned} \Phi(R^*R) = 0 &\implies \Phi(\Phi^\theta(R^*R)) = 0 \\ &\implies \pi_T(\Phi(\Phi^\theta(R^*R))) = 0 \\ &\implies \Delta \circ \pi_T(\Phi^\theta(R^*R)) = 0 \\ &\implies \pi_T(\Phi^\theta(R^*R)) = 0 \text{ because } \Delta \text{ is faithful} \\ &\implies \Phi^\theta(R^*R) = 0 \text{ by Lemma A.6} \\ &\implies R = 0 \text{ because } \Phi^\theta \text{ is faithful.} \end{aligned}$$

Thus Φ is faithful and (G, P) is amenable. \square

Corollary A.7. The Toeplitz representation $\pi_T : C^*(G, P) \rightarrow \mathcal{T}(G, P)$ is faithful.

Proof. Since (G, P) is amenable by Theorem A.1, π_T is faithful by [10, Corollary 3.8]. \square

REFERENCES

- [1] O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics II*, second ed., Springer-Verlag, Berlin, 1997.
- [2] N. Brownlowe, A. an Huef, M. Laca and I. Raeburn, Boundary quotients of the Toeplitz algebra of the affine semigroup over the natural numbers, *Ergodic Theory Dynam. Systems* **32** (2012), 35–62.
- [3] A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields, and Motives*, Colloquium Publications, Vol. 55, American Mathematical Society, 2008.
- [4] J. Crisp and M. Laca, On the Toeplitz algebras of right-angled and finite-type Artin groups, *J. Austral. Math. Soc.* **72** (2002), 223–245.
- [5] J. Crisp and M. Laca, Boundary quotients and ideals of Toeplitz C^* -algebras of Artin groups, *J. Funct. Anal.* **242** (2007), 127–156.
- [6] R. Exel, A. an Huef and I. Raeburn, Purely infinite simple C^* -algebras associated to integer dilation matrices, *Indiana Univ. Math. J.* **60** (2011), 1033–1058.
- [7] A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on the C^* -algebras of finite graphs, *J. Math. Anal. Appl.* **405** (2013), 388–399.
- [8] E. Kaniuth and K.F. Taylor, *Induced Representations of Locally Compact Groups*, Cambridge Tracts in Mathematics, vol. 197, Cambridge Univ. Press, 2013.
- [9] A. Kumjian and D. Pask, Higher rank graph C^* -algebras, *New York J. Math.* **6** (2000), 1–20.
- [10] M. Laca and I. Raeburn, Semigroup crossed products and the Toeplitz algebras of nonabelian groups, *J. Funct. Anal.* **139** (1996), 415–440.
- [11] M. Laca and I. Raeburn, Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers, *Adv. Math.* **225** (2010), 643–688.
- [12] M. Laca, I. Raeburn and J. Ramagge, Phase transition on Exel crossed products associated to dilation matrices, *J. Funct. Anal.* **261** (2011), 3633–3664.
- [13] R.C. Lyndon and P.E. Schupp, *Combinatorial Group Theory*, Springer-Verlag, Berlin, 1977.
- [14] G.J. Murphy, *C^* -Algebras and Operator Theory*, Academic Press, Boston, 1990.
- [15] A. Nica, C^* -algebras generated by isometries and Wiener-Hopf operators, *J. Operator Theory* **27** (1992), 17–52.
- [16] G.K. Pedersen, *C^* -Algebras and their Automorphism Groups*, London Math. Soc. Monographs, vol. 14, Academic Press, London, 1979.
- [17] I. Raeburn, A. Sims and T. Yeend, The C^* -algebras of finitely aligned higher-rank graphs, *J. Funct. Anal.* **213** (2004), 206–240.
- [18] I. Raeburn and D.P. Williams, *Morita Equivalence and Continuous-Trace C^* -Algebras*, Math. Surveys and Monographs, vol. 60, Amer. Math. Soc., Providence, 1998.
- [19] J. Spielberg, C^* -algebras for categories of paths associated to the Baumslag-Solitar groups, *J. London Math. Soc.* **86** (2012), 728–754.
- [20] J. Spielberg, Groupoids and C^* -algebras for categories of paths, *Trans. Amer. Math. Soc.* **366** (2014), 5771–5819.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTAGO, PO BOX 56, DUNEDIN 9054, NEW ZEALAND.

E-mail address: {lclark, astrid, iraeurn}@maths.otago.ac.nz